Eigenfunction statistics of complex systems: A common mathematical formulation

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We derive a common mathematical formulation for the eigenfunction statistics of Hermitian operators, represented by a multiparametric probability density. The system information in the formulation enters through two parameters only, namely, system size and the complexity parameter, a function of all system parameters including size. The behavior is contrary to the eigenvalue statistics which is sensitive to the complexity parameter only and shows a single parametric scaling. The existence of a mathematical formulation of both eigenfunctions and eigenvalues common to a wide range of complex systems indicates the possibility of a similar formulation for many physical properties. This also suggests the possibility to classify them in various universality classes defined by the complexity parameter.

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I. INTRODUCTION

The eigenfunction correlations of various generators of dynamics contain a wealth of information about the system, e.g., localized or delocalized nature of the dynamics, decay rate, etc. Recently the correlations were shown to be relevant for description of fluctuations of physical properties, e.g., conductance in mesoscopic systems and peak-height statistics in the Coulomb blockade regime of quantum dots [1,2]. The correlations may vary from level to level or fluctuate in different realizations of a complex system. Strong fluctuations of eigenfunctions are already known to be the hallmark of many critical point studies, e.g., the metal-insulator transition in disordered systems [3], spin glasses [4], and stock market fluctuations [5], etc. Recent studies have revealed the existence of the fluctuations in a wider range of complex systems, e.g., in the area of quantum information, nanotechnology [2], and complex networks etc. [6]. As a consequence, detailed information about the eigenfunction statistics of complex systems is very important and desirable.

During recent years, many attempts have been made to statistically formulate the eigenfunction correlations of complex systems; see, for example, [1,3,7–19]. One of the main tools used in this context is the random matrix approach which can briefly be described as follows (see [8] for details). The presence of complicated interactions (among its various subunits) in the system under investigation often makes it impossible to exactly determine the relevant operator, e.g., in a matrix representation. The elements (some or all) of the operator in the representation can then be best described by a probability distribution. This permits one to replace the operator by an ensemble of the operators which is supposed to describe the generic properties and is referred as the random matrix model of the operator. (In this paper, we focus on systems where complicated interactions, of any origin, lead to a partial or full randomization of the operator, thus allowing one to use a random matrix approach.)

The choice of an appropriate random matrix model for a system is sensitive to its physical conditions (i.e., nature and degree of interactions in various subunits, symmetry and topological conditions, dimensionality, etc.). This is because the distribution parameters of each matrix element depend on the interactions between related basis states (or parts of the system) which in turn are governed by the system conditions. In the past, this has motivated the introduction of a variety of random matrix ensembles as models for a wide range of complex systems, e.g., nuclei, atoms, molecules, disordered and chaotic systems, quantum chromodynamics, elastomechanics, electrodynamics (see the reviews [1,8,9,20–25] and references therein for details), mathematical areas such as the Riemann ζ function, enumeration problems in geometry and fluctuations in random permutations [26], biological systems [27], stock markets [5], atmospheric sciences [28], etc. [29]. For example, systems with delocalized wave dynamics (extended throughout the system) and antiunitary symmetries can be well modeled by Wigner-Dyson ensembles; the latter are Hermitian ensembles with Gaussian distributed matrix elements, with the ratio of diagonal to off-diagonal variance $\alpha=2$, (originally introduced by Cartan [30]; later developed by Wigner and Dyson to model compound nuclei and other systems) [8,9]. Cases with partially violated antiunitary symmetries can be well modeled by Dyson Brownian ensembles (BEs) [8,31] (see Sec. VI also). Ensembles with arbitrary $\alpha(\neq 2)$ [32], banded matrices [22,33,34] (elements with nonzero variance within a band around the main diagonal) and sparse matrices (with many elements with zero variance) have been successfully used to model statistical properties of the energy levels and eigenfunctions of systems with localized wave dynamics (e.g., quasi-one-dimensional wires and disordered systems of higher dimensions, chiral systems) [3,10,20,35,36]. During the last decade, many new ensembles were introduced to model systems with unitary symmetries, e.g., block form matrices for the cases with parity violation and precompound nuclei, chiral ensembles for systems with chiral symmetry in quantum chromodynamics [9,24], ensembles corresponding to Cartan's symmetric spaces, of types C, CI, D, and DIII, for normal-conducting mesoscopic systems in contact with a superconductor, and, for quasiparticles in a disordered superconductor (within Bogoliubov–de Gennes formalism) [9,37]. The non-Hermitian operators, e.g., scattering matrices [1,8,25], transfer matrices [25], or correlation matrices (appearing in time-series analysis, e.g., stock market [5], brain [27], and atmospheric studies [28]) can similarly be modeled by circu-

lar ensembles [1,8], Ginibre ensembles [38], and their more generic forms [39,40]. (The breadth of the subject is such that it is not possible to give a detailed account of all ensembles or include all references here.)

The applicability of random matrix ensembles to complex systems has been under investigation for the past few decades. The validity of the models, however, has been extensively verified in the context of eigenvalue fluctuation only; see the reviews [1,8,9,20,21,24,27,28,41–43] and references therein. The validity in the domain of eigenfunction fluctuations has so far mostly been studied either in the ergodic regime of the wave functions (see [1,9,12–14,16–19,44] for some original papers and reviews) or for quasi-one-dimensional systems [10] and specific cases, e.g., disordered systems [3]. The growing technological demands as well as the observations of hitherto unknown features among eigenfunctions (e.g., multifractal structures at critical points) of a wide range of complex systems (for example, see [3,4,35,45,46]) have made it imperative to seek statistical information in higher dimensions and beyond the ergodic regime. This motivates us to pursue the present study. It is also desirable to explore the possibilities of any connection among the critical point behavior of the eigenfunctions of different complex systems. One way to show the connection is by describing their various measures by a common mathematical formulation if possible. A recent study [47–49], has shown the existence of a similar formulation for the case of level statistics where the system information enters through a single parameter, basically a function of all system parameters. The well-known connection between the statistics of eigenfunctions and eigenvalues in the nonergodic regime [3] motivates us to seek a similar formulation for the eigenfunctions too. Such a formulation can also be useful in deriving the measure of one complex system from another.

The paper is organized as follows. Section II contains a brief revision of the single-parametric formulation of the multiparametric probability density of matrix elements for a wide range of complex systems (see [47] for details). Section III describes the derivation of the complexity-parameter-governed diffusion equation for the eigenfunction components (of the same eigenfunction or different ones), which is used in Sec. IV to study the distribution of some of the important fluctuation measures. The other measures can also be derived following the same route. The diffusion approach seems to complicate the calculation by introducing a dependence on the initial conditions, but, as discussed in Sec. IV, the statistics of the system can be recovered by integrating over all physically allowed initial conditions. The approach has an extra advantage: it provides a common analytical base for systems which can be modeled by our ensemble [given below by Eqs. (1) and (2)]. Section IV briefly discusses the role of the complexity parameter in various transitions induced due to change in system specifics. Section V contains details of the numerical verification of our analytical claims. We conclude in Sec. VI by summarizing our main results and their potential applications.

II. SINGLE-PARAMETRIC FORMULATION OF THE MATRIX ELEMENT PROBABILITY DENSITY

The eigenvalues and eigenfunctions of an operator, say H, of a system can be obtained by solving the eigenvalue equation $HU_i=\lambda_i U_i$ (with U_i and λ_i as the eigenfunction and corresponding eigenvalue, respectively) and any other physical information can then be deduced, in principle, from this knowledge. In the case of a complex system, however, the exact form of an operator, e.g., the Hamiltonian, may not be known or it may be far too complicated to solve. To deal with such a situation, one has to make a statistical hypothesis, known as the maximum entropy hypothesis, for H [50]: a sufficiently complicated system can be described by a matrix which is as random as possible under conditions compatible with the nature of the dynamics as well as the symmetry requirements. Thus, if the symmetries and the nature of the operation are approximately known in a basis space preserving the symmetries, it can be modeled by an ensemble of full or sparse random matrices in that basis. For example, an equal probability of dynamics in each region of a specific space suggests a uniform spread of the eigenfunctions in the entire associated basis space. The operator in such a basis will therefore be a full matrix, $\langle k|H|l\rangle = \Sigma_i \lambda_i U_{ki} U_{li}$, being of the same order for all combinations of basis vectors $|k\rangle, |l\rangle$ (with U_{ki} as the kth component of the eigenvector U_i). On the other hand, the dynamics localized in a space leads to variation of the eigenfunction intensities in the associated basis and the operator will be a sparse matrix.

It is clear from the above that, unlike eigenvalues, the eigenfunction statistics depends on the basis in which the matrix is represented. The knowledge, however, is still relevant because (i) it can provide important information about the system dynamics in a given basis space of interest, and (ii) it is also possible to define a relevant basis to represent an operator: it is the basis in which the constraints on the operator appear in a natural way. For example, for time-reversal-invariant systems with integer angular momentum, the relevant basis is the one in which their Hamiltonians are simultaneously expressed as real symmetric matrices [23].

In this paper, we consider a prototype distribution that can model a wide range of complex systems, namely, an ensemble of $N\times N$ Hermitian matrices H, described by a Gaussian probability density

$$\rho(H,h,b) = C \exp\left(-\sum_{s=1}^{\beta} \sum_{k,l=1;k\leqslant l}^{N} [1/2h_{kl;s}](H_{kl;s} - b_{kl;s})^2 \right). \tag{1}$$

Here $|k\rangle, |l\rangle$ are unit vectors of the arbitrary basis of size N, chosen to represent H with $H_{kl} \equiv \langle k|H|l\rangle$. The subscript s refers to the components of H_{kl}, β is their total number ($\beta=1$ for the real variable, $\beta=2$ for the complex one), C is the normalization constant, h is the variance matrix with $h_{kl;s} = \langle H^2_{kl;s}\rangle$, and b is the mean value matrix with $\langle H_{kl;s}\rangle = b_{kl;s}$. Our choice of the Hermitian nature of the ensemble restricts the present discussion to the class of systems with conservative dynamics. Following the maximum entropy hypothesis, the above ensemble can well describe the distribution of the operators for which the average behavior of the matrix elements and their variances is known. Based on the complexity of the system, the elements of the parametric matrices h, b can have various functional forms, e.g., exponential, power

law, etc. For example, the limit $h_{kl;s}\to 0$, corresponds to a nonrandom nature of H_{kl}. The limit $h_{kl;s}\to\gamma$, $b_{kl;s}\to 0$ for all $\{k,l,s\}$ gives the density for a Wigner-Dyson ensemble [8]: $\rho(H)\propto e^{-\mathrm{Tr}\,H^2}$. The limit $h_{kl;s}\to[\alpha_d\delta_{kl}+\alpha_o(1-\delta_{kl})]$, $b_{kl;s}\to 0$ for all $\{k,l,s\}$ gives the density for a Rosenzweig-Porter ensemble [32] (note that Brownian ensembles have the same density form too; see Sec. VI). A band matrix ensemble [10,20,22,33,34] with Gaussian distributed matrix elements and bandwidth t can be obtained by substituting $h_{kl;s}\to 0$, $b_{kl;s}\to 0$ if $|k-l|>t$ and $h_{kl;s}\propto a(|k-l|)$ for $|k-l|\leqslant t$ with various possible forms of the function, a, e.g., exponential, rectangular, etc. Similarly other ensembles with uncorrelated matrix elements, some of them with Gaussian randomness and others nonrandom, can be represented by appropriate choice of h and b parameters [47,48].

Equation (1) is applicable for cases with mutually independent matrix elements, with no condition imposed on their moments higher than second order. Here we briefly mention only two such cases, namely, disordered systems and mixed dynamical systems (the application to other cases, e.g., algebraic [26,51] or algorithmic complexity [51], will be discussed elsewhere). During the recent past, specific cases of Eq. (1) have been extensively used to model the energy level statistics of disordered systems within the independent electron approximation (the latter results in independent matrix elements of the Hamiltonian). One such example is the power law random banded matrix (PRBM) ensemble (each h_{kl} with a power law dependence on the distance from the diagonal) [34] which has been shown to be a good model for the level-statistics of the Anderson Hamiltonian (AH) [3]. The ensemble (1) was also used recently to prove, analytically as well as numerically, the single-parameter scaling of the level statistics of the Anderson Hamiltonian and its mapping to single-parametric Brownian ensembles [48]. (Later in this paper, the AH, BE, and PRBM cases are also used to verify our analytical predictions for the eigenfunction statistics.)

Another potential application of Eq. (1) is to systems with mixed dynamics where, similar to disorder, Kolmogorov-Arnold-Moser (KAM) tori lead to a localization of dynamics [52]. The connection of quantum systems in classically chaotic and integrable regimes to Wigner-Dyson ensembles and Poisson ensembles, respectively, is already well established [23,41,53–55]. In the past, it has been suggested that a mixed Hamiltonian (or time-evolution operator) in a relevant basis should appear as a block diagonal matrix, each block being associated with an isolated region of the classical phase space [33,23,53]. In cases where a chaotic region can be decomposed in nearly but not completely isolated subregions, blocks are expected to be connected through small but nonzero matrix elements. The average size of these matrix elements, i.e., the quantum constraints, will be related to the flux connecting different regions, i.e., the classical information. We further suggest that the regimes with stable islands can be modeled by blocks with nonrandom elements (e.g., zero variance and nonzero mean). Chaotic regimes can be modeled by blocks with randomly distributed elements (e.g., the same nonzero variance for all elements or only within a band). Based on the nature of the dynamics, a chaotic block may further contain a hierarchy of random and nonrandom subblocks; various diagonal blocks may also be correlated. Equation (1) can then be applied to model the Hamiltonian by choosing the matrix element variances appropriate to the block in which they appear.

Equation (1) cannot serve as a good model for cases with correlated matrix elements. For example, particle-particle interactions in nuclei [1,7,56,21] and electron-electron interactions in disordered systems can lead to correlations among elements of the Hamiltonian [1,49]; the correlation coefficients depend on various system parameters. In general, such cases can occur when the interaction (described by H) between any two basis states is influenced by the other states. In the past, consideration of particle correlations in nuclei led to the introduction of embedded ensembles [7,9,56]; however, no significant progress has been made so far in dealing analytically with these ensembles.

In general, an increase of constraints on the system dynamics subjects higher moments of the matrix elements to certain specific conditions. This motivates us to consider an alternative ensemble, namely, the maximum entropy ensemble with restricted higher moments. Within the maximum entropy hypothesis, the probability density for such cases turns out to be non-Gaussian [49]: $\tilde{\rho}(H)=C\rho(H)$ where

$$\rho(H)=\prod_{s=1}^{\beta}\prod_{r=1}^{n}\exp\left[-\sum_{p(r)}b_{p(r)}\left(\prod_{i_pj_p}^{r}H_{i_pj_p;s}\right)\right] \tag{2}$$

with C as a normalization constant. Here each H_{jk} is expressed in terms of its β components, ($\beta=1$ for the real symmetric matrices and $\beta=2$ for the complex Hermitian case): $H_{jk}\equiv\Sigma_{s=1}^{\beta}(i)^{s-1}H_{jk;s}$. Here the symbol $p(r)$ refers to a combination of r matrix elements chosen from a total set of $\tilde{M}=N(N+1)/2$ of them; note the terms present in a given combination need not be all different. The $\Pi^{r}_{i_pj_p}$ implies a product over r terms present in the pth combination with coefficient $b_{p(r)}$ as a measure of their correlation: $\langle\Pi^{r}_{i_pj_p}H_{i_pj_p;s}\rangle=\partial\log C/\partial b_{p(r)}$. The $\Sigma_{p(r)}$ is a sum over all possible combinations [total $(\tilde{M})^r$] of r elements chosen from a total set of $\tilde{M}=N(N+1)/2$ of them.

The potential use of Eq. (2) to disordered systems with e-e interaction is discussed in [49]. Here we briefly discuss a few more examples. Systems with chiral symmetry can be modeled by Hermitian ensembles with block form matrices $H=\left(\begin{smallmatrix}0 & W\\ W^{+} & 0\end{smallmatrix}\right)$, with W as a matrix of size N. Here, since $H_{k',l}=H^{*}_{l,k'}$ for $1\geqslant k,l\geqslant N$ (with $k'\equiv k+N, l'\equiv l+N$), the correlations between these elements are subjected to the conditions

$$\langle H_{k',l}H_{l,k'}\rangle=\langle|H_{k',l}|^2\rangle=\langle|H_{l,k'}|^2\rangle,$$

$$\langle H_{k,l}\rangle=\langle H_{k',l'}\rangle. \tag{3}$$

However, because $H_{k,l}=H_{k',l'}=0$, all other matrix elements are uncorrelated. For a simple explanation, let us restrict attention to the case of a real matrix W with Gaussian distributed elements. The ensemble can then be represented by Eq. (2) with $n=2$ or equivalently by the density

$$\rho(H,a,b) = C \exp\left(-\sum_{i\leq j,k\leq l} b_{ijkl} H_{ij} H_{kl} - \sum_{kl} a_{kl} H_{kl}\right), \quad (4)$$

with the following conditions on a and b:

$$b_{k',l,l,k'} = b_{k',l,k',l} = b_{l,k',l,k'},$$

$$a_{kl} = a_{k',l'} = 0, \quad a_{k',l} = a_{l,k'}. \quad (5)$$

Note that b parameters corresponding to other pairs of elements (both or at least one in diagonal blocks) diverge due to zero correlation between elements in such pairs. The cases with other types of correlated blocks can similarly be modeled by applying appropriate conditions on the b parameters, which correspond to combinations of matrix elements appearing in opposite blocks. For example, the ensemble C describes the cases with particle-hole symmetry with a Hamiltonian $H=\begin{pmatrix} A & B \\ B^{+} & -A^{T} \end{pmatrix}$ (see [37] for details). Now the correlations between various elements must be subjected to the conditions $H_{k',l}=H^{*}_{l,k'}$, $H_{k,l}=-H_{l',k'}$; this implies another set of nonzero correlations [besides those given by Eq. (3)]: $\langle H_{k,l} H_{l',k'}\rangle = -\langle |H_{k,l}|^2\rangle = -\langle |H_{l',k'}|^2\rangle$. For Gaussian distributed real matrices A and B, the case can again be modeled by Eq. (4); however, now the b parameters for other pairs [besides those given in Eq. (5)] can also be finite and satisfy the equality: (i) $\alpha b_{k,l,l',k'} = b_{k,l,k,l} = b_{l',k',l',k'}$; (ii) $b_{k,l,k',l} = b_{k,l,l,k'} = \alpha b_{l',k',l,k'} = \alpha b_{l',k',k',l}$ with $\alpha=-1$.

Recently it was shown [47,49] that the distribution ρ for both cases [Eqs. (1) and (2)] appear as the nonequilibrium stages of a Brownian-type diffusion process in the matrix space, evolving with respect to a single parameter which is a function of the distribution parameters of the ensemble:

$$\frac{\partial \rho}{\partial Y} = L_{+}\rho \quad (6)$$

with

$$L_{\pm} = \sum_{k,l;s} \frac{\partial}{\partial H_{kl;s}} \left(\frac{g_{kl}}{2} \frac{\partial}{\partial H_{kl;s}} \pm \gamma H_{kl;s} \right) \quad (7)$$

where $g_{kl}=1+\delta_{kl}$. The variable Y is the parameter governing the evolution of matrix elements subjected to various system conditions. For the case (1),

$$Y = -\frac{1}{2M\gamma} \ln\left(\prod_{k\leq l}{}' \prod_{s=1}^{\beta} |x_{kl;s}| |b_{kl;s}|^2 \right) + C_0 \quad (8)$$

with Π' implying a product over nonzero $b_{kl;s}$ and $x_{kl;s} = 1-(2-\delta_{kl})\gamma h_{kl;s}$; C_0 is a constant determined by the initial distribution and M is the number of all nonzero parameters $x_{kl;s}$ and $b_{kl;s}$. The parameter γ is arbitrary, giving the freedom to choose the end of the evolution; $\lim h_{kl;s} \to \gamma$, $b_{kl;s} \to 0$ for all k, l gives $Y \to \infty$ and the steady state (a Wigner-Dyson ensemble). The distribution parameters being indicators of the complexity of the system, Y can be termed the complexity parameter [47]. Some examples of the calculation of Y from Eq. (8) are discussed in [47] (for banded ensembles) and in [48] (for the Anderson Hamiltonian). Y in the case of a mixed system can similarly be calculated if one knows the details of the mixed dynamics.

In general, the form of parameter Y for Eq. (2) is quite complicated; its details can be found in [49]. However, for case (4), which is the Gaussian version of case (2), Y can be given as

$$Y = \sum_{kl} \int da_{kl} X + \sum_{ijkl} \int db_{ijkl} X + \text{const} \quad (9)$$

where summation is implied over the distribution parameters with finite values only, and $X=(\Sigma_{kl} f_{kl} + \Sigma_{ijkl} f_{ijkl})^{-1}$ with $f_{kl} = \gamma a_{kl} - 2(\Sigma_{mn} a_{nm} b_{klmn} + a_{mn} b_{klnm})$, $f_{ijkl} = (\gamma b_{ijkl} - 2\Sigma_{mn} b_{ijnm} b_{klmn})$. For further clarification we refer the reader to [49] where an example, namely, the modeling of the lowest Landau level of a disordered quantum Hall system by Eq. (4) and calculation of the corresponding Y is discussed.

It is easy to solve Eq. (6) for arbitrary initial conditions, say H_0 at $Y=Y_0$,: $\rho(H,Y|H_0,Y_0) \propto \exp[-(\alpha/2)\mathrm{Tr}(H-\eta H_0)^2]$ with $\alpha=\gamma(1-\eta^2)^{-1}$ and $\eta=e^{-\gamma Y}$. The probability density of H can now be extracted by integrating over an ensemble of initial conditions: $\rho(H,Y-Y_0) = \int \rho(H,Y|H_0,Y_0) \times \rho(H_0,Y_0) dH_0$. It is often useful to study the statistics of the perturbed Hamiltonian H in the eigenfunction basis of unperturbed Hamiltonian H_0. Thus if the eigenfunctions of H_0 are chosen as the basis vectors $|k\rangle, |l\rangle$ etc., and the initial distribution is given by $\rho(H_0) \propto \exp[-(1/2)\Sigma_j H^2_{0;jj}]$, the eigenvalue equation $UH=\Lambda U$ can be used to transform $\rho(H)$ from matrix space to eigenvalue-eigenvector space $\{\lambda, U\}$:

$$\rho(H,Y) \propto \prod_{k,l;k<l}^{N} |\lambda_k - \lambda_l|^{\beta} \exp\left(-(\gamma/2) \sum_{j=1}^{N} \lambda_j^2 - (\gamma\mu/2) \times \sum_{k<l} |\lambda_k - \lambda_l|^2 |U_{jk}|^2 |U_{jl}|^2 \right) \quad (10)$$

where $\mu=(e^{2\gamma(Y-Y_0)}-1)^{-1}$.

As indicated by Eqs. (6) and (10), the ensemble densities for various complex systems (i.e., different h,b matrices) undergo a similar evolution as a function of Y. The Y-governed flow for the joint distribution of the desired eigenfunction components and eigenvalues can be obtained, in principle, by integrating either Eq. (6) or Eq. (10) over all the undesired ones; however, it is easier to integrate Eq. (6). To explain the technique, we consider some of the important cases in this paper.

III. DIFFUSION EQUATION FOR EIGENFUNCTION COMPONENTS AND RELATED EIGENVALUES

The kth component U_{kl} of an eigenstate U_l is a measure of the contribution of the kth basis state to the eigenstate. Experimental observations of complex systems indicate the level-to-level variations as well as sample-to-sample fluctuations of the contribution. As a result, the knowledge of the average behavior of the components is not enough and one needs to study their distribution. In this section, we consider the joint probability distributions of a few relevant combinations of the components of the operator H. The basis chosen for the representation of the eigenfunctions is the one in which the matrix elements of H have distribution (1) [or (2)].

We use the following notation in reference to various correlations. For a joint distribution P_{rs}, the subscripts r and s refer to the number of components of each eigenvector and the number of eigenvectors considered, respectively. For example, for a joint distribution of n components of m eigenvectors along with their eigenvalues, $r=n$ and $s=m$.

A. Joint distribution of a given component of all eigenfunctions and eigenvalues

It is often relevant to know the influence of a particular basis state on the system dynamics at various energies and with varying complexity of the system. The information can be obtained by a knowledge of the distribution of the same component of various eigenfunctions and its Y governed evolution. For example, let us calculate the joint distribution of a given component of all eigenvectors and the eigenvalues. Let $P_{1N}(Z,E,Y)$ be the probability, at a given Y, of finding the jth component U_{jn} of the eigenfunctions U_n of H between z_{jn} and $z_{jn}+dz_{jn}$ and the eigenvalues λ_n between e_n and e_n+de_n for $n=1\rightarrow N$ (with $Z\equiv\{z_{jn}\}$, $E\equiv\{e_n\}$). It can be expressed as an average over the entire ensemble ρ:

$$P_{1N}(Z,E,Y)=\int f_N(Z,E,U,\lambda)\rho(H,Y)dH \tag{11}$$

with $f_r(Z,E,U,\lambda)=\Pi_{n=1}^{r}\delta(z_{jn}-U_{jn})\delta^{\beta-1}(z_{jn}^*-U_{jn}^*)\delta(e_n-\lambda_n)$. The Y-dependent evolution equation for P_{1N} can now be derived by connecting the parametric derivatives of P_{1N} to its derivatives with respect to eigenvectors. The steps can briefly be described as follows. As the Y dependence of P_{1N} comes only through ρ, one can write

$$\frac{\partial P_{1N}}{\partial Y}=\int dH\, f_N L_+\rho=\int d\,H\rho L_- f_N+\tilde{\gamma}P_{1N} \tag{12}$$

with $\tilde{\gamma}=\beta N(N+1)\gamma/2$. Equation (12) is obtained, first, by differentiating Eq. (11) with respect to Y, then using Eq. (6), followed by partial integration. Due to the δ-function nature of f_N, its derivatives with respect to matrix elements can further be reduced to the derivatives with respect to Z and E,

$$\frac{\partial f_N}{\partial H_{kl;s}}=-\sum_{n=1}^{N}\left(\frac{\partial\lambda_n}{\partial H_{kl;s}}\frac{\partial f_N}{\partial e_n}+\frac{\partial U_{jn}}{\partial H_{kl;s}}\frac{\partial f_N}{\partial z_{jn}}+\frac{\partial U_{jn}^*}{\partial H_{kl;s}}\frac{\partial f_N}{\partial z_{jn}^*}\right). \tag{13}$$

The second derivative of f_N can now be obtained from Eq. (13) (see [47]). The substitution of Eq. (13) in Eq. (12) helps as the derivatives with respect to z_{jn} and e_n can be taken out of the integral. It can further be simplified by a knowledge of the effect of a small perturbation of H on its eigenvalues and eigenvectors; the related results are given in Appendix B (see [47] for the details). Using the relations, Eq. (12) can be rewritten as

$$\frac{\partial P}{\partial Y}=(L_Z+L_Z^*)P+L_E P \tag{14}$$

where $P=C_1P_{1N}$, $C_1=e^{-\tilde{\gamma}Y}$, and

$$L_Z=\frac{\beta^2}{4}\sum_{n,m=1;n\neq m}^{N}\frac{1}{(e_n-e_m)^2}\frac{\partial}{\partial z_{jn}}\times\left(\frac{\partial}{\partial z_{jn}^*}|z_{jm}|^2-\frac{\partial}{\partial z_{jm}}z_{jn}z_{jm}+z_{jn}\right),$$

$$L_E=\sum_n\frac{\partial}{\partial e_n}\left(\gamma e_n+\sum_{m;m\neq n}\frac{\beta}{e_m-e_n}+\frac{\partial}{\partial e_n}\right), \tag{15}$$

where L_Z^* implies the complex conjugate of L_Z; note that $L_Z=L_Z^*$ for the $\beta=1$ case. Equation (14) describes the Y-governed diffusion of a given component of all eigenvectors and all eigenvalues. Its solution depends on the choice of initial condition H_0. In the diagonal representation of H_0, taking $\rho(H_0)\propto\exp[-(1/2)\Sigma_j H_{0;jj}^2]$, the solution can be given as

$$P_{1N}\propto\prod_{k,l;k<l}^{N}|e_k-e_l|^\beta\exp\left(-(1/2)\sum_{j=1}^{N}e_j^2-(\mu/2)\times\sum_{m<n}|e_n-e_m|^2|U_{jn}|^2|U_{jm}|^2\right) \tag{16}$$

with μ the same as in Eq. (10). [Note that the above result can directly be obtained from Eq. (10) too.]

B. Joint distribution of all components of A given the eigenfunction and its eigenvalue

The distribution of the components of a specific eigenstate contains information about various basis states contributing to the state, which in turn determines its spread. Proceeding along the same lines as for P_{1N}, the diffusion equation for the joint probability P_{N1} of the components U_{nk}, $n=1\rightarrow N$, of an eigenvector U_k and the corresponding eigenvalue λ_k can also be obtained. The evolution of

$$P_{N1}(Z_k,e_k,Y)=\int\tilde{f}_k\rho(H,Y)dH, \tag{17}$$

with $\tilde{f}_k=\delta(Z_k-U_k)\delta^{\beta-1}(Z_k^*-U_k^*)\delta(e_k-\lambda_k)$, can again be shown to be described by

$$\frac{\partial P_{N1}}{\partial Y}=F_k+F_k^*+L_{e_k}P_{N1} \tag{18}$$

where $F_k=(\beta^2/4)\Sigma_{q=1}^{2}L_{qk}$ with

$$L_{1k}=\sum_{n=1}^{N}\frac{\partial}{\partial z_{nk}}(z_{nk}Q_{nn;k}^{02}),$$

$$L_{2k}=\sum_{m,n=1}^{N}\frac{\partial^2}{\partial z_{nk}\partial z_{mk}^*}Q_{mn;k}^{12},$$

$$L_{e_k}P_{N1}=\frac{\partial}{\partial e_k}\left(\gamma e_k P+\frac{\partial P}{\partial e_k}+\beta Q_{nn;k}^{01}\right), \tag{19}$$

and

$$Q^{rs}_{mn;k} = \sum_{j;j\neq k} \int \frac{(z_{nj}z^*_{mj})^r}{(e_k - e_j)^s} P_{N2} d\tau_j. \quad (20)$$

Here $d\tau_j \equiv de_j d^\beta Z_j$ with $d^\beta Z \equiv dZ\, dZ^*$ and $P_{N2} = P_{N2}(Z_k, Z_j, e_k, e_j)$ is the joint probability of all components of the two eigenvectors $Z_j \equiv \{z_{nj}\}$ and $Z_k \equiv \{z_{nk}\}$ $(n=1\rightarrow N)$ along with their eigenvalues e_j and e_k, respectively,

$$P_{N2} = \int \tilde{f}_k \tilde{f}_j \rho(H,Y) dH. \quad (21)$$

The presence of eigenvalue-eigenfunction correlations in the exponent of $\rho(H)$ [e.g., Eq. (10)] as well as the terms of type $(e_j - e_k)^{-2}$ in the denominator of Eq. (20) makes it difficult to write F_k [in Eq. (18)] as a function of $P_{N1}(Z_k, e_k)$. To write Eq. (18) in a closed form, it is necessary to approximate $Q^{rs}_{mn;k}$ (Appendix A):

$$Q^{rs}_{mn;k} \approx D_k^{-s}(N-1)^{1-r}\chi^{s/2}(\delta_{mn} - z^*_{mk}z_{nk})^r P_{N1} \quad (22)$$

with D_k as the local mean level spacing at energy e_k. Here $\chi=1$ for $\mu < \zeta_k^d$, $\chi \sim (\mu/\zeta_k^d)$ for $\mu > \zeta_k^d$ with ζ as the ensemble averaged localization length of the eigenfunction U_k and d as the system dimension. The length ζ enters in the formulation due to its relation with the typical intensity of a wave function: $|U_{nk}|^2 \equiv |z_{nk}|^2 \sim \zeta_k^{-d}$.

The substitution of Eq. (22) in Eq. (19) helps to write F_k in terms of P_{N1}, thus reducing the evolution equation (18) for P_{N1} to a closed form:

$$F_k = \frac{\beta^2}{4D^2} \sum_{n=1}^{N} \frac{\partial}{\partial z_{nk}} \left(\sum_m \frac{\partial h_2}{\partial z^*_{mk}} + h_1 \right) \quad (23)$$

with $h_1 = (N-1)\chi z_{nk} P_{N1}$, $h_2 = \chi(\delta_{mn} - z_{nk}z^*_{mk})P_{N1}$.

C. Joint distribution of all components of q eigenfunctions and their eigenvalues

For certain physical properties, e.g., susceptibility, a knowledge of the correlations among two (or more) eigenvectors at two different space points may be required. The fluctuations of such correlations can be determined by the joint probability density P_{Nq} of the components U_{nk} $(n=1\rightarrow N)$ of q eigenvectors U_k $(k=1\rightarrow q)$ where

$$P_{Nq}(Z_1, Z_2, \ldots, Z_q, Y) = \int \prod_{k=1}^{q} \tilde{f}_k \rho(H,Y) dH. \quad (24)$$

Proceeding exactly as in the previous two cases, the Y-governed diffusion of P_{Nq} can be shown to be described as

$$\frac{\partial P_{Nq}}{\partial Y} = \sum_{k=1}^{q} (\tilde{F}_k + \tilde{F}^*_k + L_{e_k} P_{Nq}) \quad (25)$$

where

$$\tilde{F}_k = F_k + \frac{\beta^2}{4} \sum_{l=1;\neq k}^{q} \sum_{m,n=1}^{N} \frac{\partial^2}{\partial z_{nk}\partial z_{ml}} \left(\frac{z_{nk}z_{ml}}{(e_k - e_l)^2} \right) P_{Nq}. \quad (26)$$

Note that although F_k, L_{1k}, L_{2k} are still defined as in Eqs. (18) and (19) the definition of Q is now slightly altered with $P_{N(q+1)}$ replacing P_{N2} in Eq. (20). Here $P_{N(q+1)}$ is the joint probability density of $q+1$ eigenfunctions, namely, $Z_1, Z_2, \ldots, Z_q$ along with Z_j (with $j>q$). As in the previous case, the integral Q can again be approximated so as to express F_k in terms of P_{Nq}: $Q^{rs}_{mn;k} \approx D_k^{-s}(N-1)^{1-r}\chi^{s/2}(\delta_{mn} - \Sigma_{l=1}^{q} z^*_{ml}z_{nl})^r P_{Nq}$. Here again $\chi=1$ for $\mu < \zeta_k^d$ and $\chi \sim (\mu/\zeta^d)$ for $\mu > \zeta_k^d$.

The above approximation for Q leaves the expression for F_k in the same form as in Eq. (23); however, now $h_1 = \chi(N-1)P_{Nq}$ and $h_2 = \chi(\delta_{mn} - \Sigma_{l=1}^{q} z^*_{ml}z_{nl})P_{Nq}$. The substitution of F_k in $\tilde{F}_k$ gives the latter as a function of P_{Nq} which in turn reduces Eq. (26) to a closed form for P_{Nq}. The equation can then be used, by integrating over undesired components, to obtain the distributions of various combinations of eigenfunction components.

IV. DIFFUSION EQUATION FOR FLUCTUATION MEASURES OF EIGENFUNCTIONS

The ensemble average of any measure of the eigenfunction correlations can be expressed in terms of P ($P \propto P_{rq}$ for a correlation function of r components of q eigenstates). For example, the average of a measure, say C, describing the correlation among a set X of eigenfunction components can be written as

$$\langle C(X;Y) \rangle = \int_0^\infty C(X;Y) P(X;Y) dX, \quad (27)$$

where $\langle \cdot \rangle$ denotes an averaging over various realizations of the sample. However, the strength of the reproducible fluctuations of the correlations in different realizations of the same complex system is of the order of the averages. As a consequence, a knowledge of just the averages is not enough and it is necessary to know the distributions of correlations.

The Y-governed evolution of the distribution P_C of a measure C can be obtained by an integration of the undesired variables in Eq. (14) [or Eq. (18) or Eq. (25) as per requirements]. As examples, we derive the evolution equations for a few important measures in this section. The involved integrals are, however, quite tedious and analytical approximations seem necessary to reduce the equation to a closed form. As a check on our results, we study the $Y \rightarrow \infty$ limit of each measure. This limit corresponds to the flow of ensemble (1) [and ensemble (2)] to its steady state, that is, a Wigner-Dyson ensemble. As a consequence, each measure is expected to evolve to its Wigner-Dyson limit as $Y \rightarrow \infty$. We verify our results numerically too; the details are given in Sec. VI.

A. Distribution of local eigenfunction intensity

The distribution function of the local eigenfunction intensity, i.e., the eigenfunction intensity u at a given basis state, say n, can be defined as $P_u(u,e) = \langle \Sigma_{k=1}^{N} \delta(u - N|z_{nk}|^2)\delta(e - e_k) \rangle$. The diffusion of P_u as a function of Λ can be obtained from either Eq. (14) with $P \propto P_{1N}$ or Eq. (18). For technical simplification, however, we choose the former and first study the evolution of the distribution $P_{11}(x,e)$ of an eigenfunction

component $x=N^{1/2}z_{nk}=(u^{1/2})$ at an energy e, defined as

$$P_{11}(x,x^*,e)=\langle\delta_x^\beta\delta_e\rangle=\int\delta_x^\beta\delta_e P_{1N}(Z,E,Y)dE\,d^\beta Z, \quad (28)$$

where $\delta_x^\beta=\delta(x-\sqrt{N}z_{nk})\delta^{\beta-1}(x^*-\sqrt{N}z_{nk}^*)$, $\delta_e=\delta(e-e_k)$, and $d\tau\equiv dE\,d^\beta Z$. The diffusion equation for $P_{11}(x,e)$ can be obtained by integrating Eq. (14), with $P\propto P_{1N}$, over the variables e_j and z_{nj}, $j=1\to N$,

$$\frac{\partial P_{11}}{\partial Y}=\frac{\beta^2}{4}\left(2\frac{\partial^2 G_1}{\partial x\partial x^*}+\frac{\partial(xG_0)}{\partial x}+\frac{\partial(x^*G_0)}{\partial x^*}\right)+L_eP_{11} \quad (29)$$

where

$$G_r(x,e)\equiv\sum_{j;j\neq k}\int\delta_x^\beta\delta_e\frac{|z_{nj}|^{2r}}{(e_k-e_j)^2}P_{1N}d\tau, \quad (30)$$

with $r=0,1$ and $\int\delta_x^\beta\delta_e[L_EP_{1N}]dE\,d^\beta Z=L_eP_{11}$.

Equation (29) describes the sensitivity of the local intensity distribution to the energy scale e as well as various system parameters. As discussed in Appendix A [see Eq. (A7)], G_r can be approximated as

$$G_r\approx\mu\chi_0(N-1)^{1-r}(N-|x|^2)^rP_{11}(x)/D_k^2 \quad (31)$$

with $\chi_0=\mu^{-1}$ for $\mu|x|^2<1$ and $\chi_0\sim|x|^2$ for $\mu|x|^2>1$ where $\mu=(e^{2\gamma(Y-Y_0)}-1)^{-1}$ and D_k as the local mean level spacing at energy e_k. Substitution of the approximated G_r in Eq. (29) and integration over e gives the energy-averaged local intensity distribution $P_x(x)=\int P_{11}(x,e)de$:

$$\frac{\partial P_x}{\partial\Lambda_u}=\frac{\beta^2}{4}\left(2\frac{\partial^2[h_2(x)P_x]}{\partial x\partial x^*}+\frac{\partial[h_1(x)P_x]}{\partial x}+\frac{\partial[h_1(x^*)P_x]}{\partial x^*}\right) \quad (32)$$

with $h_2(x)=\chi_0(N-x^2)$, $h_1(x)=\chi_0(N-1)x$. Here $\Lambda_u=\mu\Lambda$ with $\Lambda=(Y-Y_0)/D_k^2$. Equation (32) suggests that the evolution of P_x is governed by a rescaled parameter Λ_u instead of Y.

For the cases $|x|^2\ll N$ (thus $\Lambda_u=\Lambda$), the above equation can easily be solved: $P_x(x,\Lambda|x_0)\propto e^{-\beta|x-\gamma x_0|^2/2(1-\gamma^2)}$ with $\gamma=e^{-\beta N\Lambda/2}$ and $P_{x_0}(x_0)$ as the initial distribution. The steady state limit $\partial P_x/\partial\Lambda\to 0$ of Eq. (32) occurs at $\Lambda\to\infty$. The solution in this limit corresponds to the Wigner-Dyson case, i.e., $P_x(x,\Lambda\to\infty)\propto e^{-\beta|x|^2/2}$ or, equivalently, the Porter-Thomas distribution $P_u(u,\Lambda\to\infty)\propto u^{(\beta-2)/2}e^{-\beta u/2}$ [1,8] [using $u=|x|^2$, which gives $P_u=P_x(2|x|)^{-1}$].

It is desirable to know the solution P_x of Eq. (32), or, alternatively, P_u for finite, nonzero Λ_u and all ranges of u. In the diagonal representation of H_0, which corresponds to an initial distribution $P_{u_0}(u_0,\Lambda=0)=N^{-1}[\delta(u-1)+(N-1)\delta(u)]$, Eq. (32) gives the following short-range behavior of P_u:

$$P_u=(\beta u/2)^{\beta/2-1}\frac{e^{-\beta u/2}}{\Gamma(\beta/2)}\left(1+\frac{\kappa}{2}[(\beta+2)/\beta-(\beta+2)\sqrt{u}+\beta u/2]+\cdots\right),\quad u\lesssim\kappa^{-1/2} \quad (33)$$

$$\approx(\beta u/2)^{\beta/2-1}\frac{1}{\Gamma(\beta/2)}\exp\left((\beta/2)\left(-u+\frac{\kappa}{2}u^2+\cdots\right)\right),$$

$$\kappa^{-1/2}\lesssim u\lesssim\kappa^{-1} \quad (34)$$

where $\kappa=e^{-2\beta N\Lambda}$ [note that $\kappa\approx\mu$ in the large-Y limit and for $D_k^2\sim(\beta N)^{-1}$].

The tail behavior of a distribution has a significant influence on its moments and the related physical properties. The asymptotic analysis of Eq. (32) shows $P_u(u)$ to be a broad distribution:

$$P_u(u)\simeq\exp\left(-\alpha_{u0}u^{1/2}-\sum_{n=1}^{M}\alpha_{un}\ln^n(\kappa u)\right),\quad u\gtrsim\kappa^{-1}. \quad (35)$$

Here the coefficients are sensitive to system specifics: $\alpha_{u0}\simeq 4q_1\beta^{-1}(e^{\beta N\Lambda_u}-1)$, $\alpha_{u1}\simeq -N/4$, $\alpha_{u2}\simeq(N\beta/16)e^{\beta N\Lambda_u}$, $\alpha_{un;n>2}\simeq(-1)^n(\nu_n\beta^2N/4)e^{2\beta N\Lambda_u}$ with ν_n decreasing as n increases. The decreasing coefficients along with alternating $\pm$ signs lead to near-cancellation of higher-order terms (with $n>2$) in the exponent. Consequently, the tail is dominated by log-normal behavior for systems with large, finite Λ strengths and a weaker than exponential decay in the $\Lambda\to 0$ limit.

Equation (35) indicates the existence of a log-normal asymptotic tail for the local eigenfunction intensity of any complex system with finite, nonzero Λ_u. A log-normal behavior of $P(u)$ suggests a power law behavior of its moments: $\langle u^q\rangle\propto N^{-d_q}$ [17]. Here d_q is an effective dimension which can be different from the spatial dimension d. The form of $P_u(u)$ at finite Λ is therefore fixed by a spectrum of scaling exponents (as the moments can be used to recreate the distribution); the situation is termed multifractal scaling. Further, as shown later, a log-normal tail of $P(u)$ results in similar behavior of the distributions of other related correlations and physical properties. Such a behavior has already been indicated for the physical properties, e.g., conductance, density of states, local density of states, and relaxation time, etc., of disordered systems [3].

The significance of the above $P(u)$ formulation is that here system dependence (other than size) enters only through one parameter, namely, Λ. This being valid for any complex system modeled by Eq. (1) [and Eq. (2)], it is thus applicable for disordered systems too. It is therefore relevant to compare our result with those obtained for disordered systems using other techniques (using the renormalization group theory approach for dimension $d=2+\epsilon$, $\epsilon<1$ [43], and by using Berezinski and Abrikosov-Ryzkhin techniques for strictly $d=1$ cases [8,11,57]; the techniques predict an $e^{-\alpha_1 u^{1/2}}$ tail for $d=1$, a log-normal tail for $d=2$, and a log-cube tail for $d=3$ case. However, our technique predicts a log-polynomial behavior dominated by the log-normal term for all dimensions.

B. Inverse participation ratio

The qth-order inverse participation ratio (IPR) I_q of an eigenvector, say U_k, is defined as $I_q(k)=\Sigma_{j=1}^N|U_{jk}|^{2q}$. The

physical meaning of I_q can be illustrated by two limiting cases: (i) an eigenfunction with identical components $U_{jk}=N^{-1/2}$ corresponds to $I_q(k)=N^{1-q}$, and, (ii) an eigenfunction with only one nonzero component (say the nth) which gives $U_{jk}=\delta_{nk}$ and $I_q(k)=1$. The case (i) corresponds to completely ergodic eigenfunctions covering randomly but uniformly the whole sample of volume V. The case (ii) corresponds to a wave function localized in the neighborhood of a single basis state. Thus I_q, in general, is related to the reciprocal of the number of components significantly different from zero and contains information about the spread of a wave function in the basis space. For example, for a d-dimensional exponentially localized state, $I_2 \sim (a/\zeta)^d$, where a and ζ are the lattice constant and localization length, respectively. Consequently, the typical value of I_2 is a frequently used characteristic of the eigenfunction localization [3]: $I_2^{typ}=\exp\langle \ln I_2\rangle \approx N^{-D_2}$ with D_2 a system-dependent scaling exponent (also known as the correlation dimension).

The ensemble average of I_q is related to the qth moment of the distribution $P_x(x)$: $\langle I_q\rangle = N^{1-q}\int_0^\infty |x|^{2q} P_x(x) d^\beta x = \int I_q P_{I_q} dI_q$. The average inverse participation ratios can therefore provide information about the scaling exponents. As a consequence, it is useful to know the effect of changing system parameters on $\langle I_q\rangle$. Due to the $P(u)$ decay for the ranges $\mu u \gtrsim 1$, the major contribution to $\langle I_q\rangle$ comes from the region $\mu u \leqslant 1$. From Eq. (32), it can be shown that

$$\frac{\partial \langle I_q\rangle}{\partial \Lambda} \approx q\alpha \langle I_{q-1}\rangle - qt\langle I_q\rangle, \tag{36}$$

where $\alpha=2q+\beta-2$, $t=N\beta+2q-2$. Equation (36) depends on two parameters, namely, Λ and t, which results in a different power law behavior for each $\langle I_q\rangle$,

$$\langle I_q(\Lambda)\rangle = e^{-qt\Lambda}\left(\langle I_q(0)\rangle + \alpha\int_0^\Lambda \langle I_{q-1}(r)\rangle e^{qtr} dr\right). \tag{37}$$

For $\Lambda \to \infty$, Eq. (37) gives a correct steady state limit, namely, Wigner-Dyson behavior: $\langle I_q\rangle \to (\alpha/t)\langle I_{q-1}\rangle$ or $\langle I_q\rangle = [(2q)!/2^q q!]N^{1-q}$ for $\beta=1$ and $\langle I_q\rangle = q! N^{1-q}$ for $\beta=2$. For finite nonzero Λ, $\langle I_q\rangle$ can be determined if $\langle I_{q-1}(\Lambda)\rangle$ as well as some past information about the system [to choose it as an initial state which will give $\langle I_q(0)\rangle$] is known. For example, for systems where completely localized wave function dynamics is a valid physical possibility (e.g., disordered systems, mixed systems, etc.), it can be chosen as the initial state which corresponds to $\langle I_q(0)\rangle = 1$; this gives $\langle I_1(\Lambda)\rangle = 1$, $\langle I_q(\Lambda)\rangle \approx e^{-q\beta N\Lambda}$ for $q<N$.

In general, the IPR fluctuations reflect the level-to-level variations of the spatial structure of eigenfunctions. In a complex system, e.g., a nanosystem, however, the sample-to-sample fluctuations of the eigenfunctions also manifest themselves through IPR fluctuations which makes knowledge of the I_q distribution over the whole ensemble of samples relevant. The distribution P_{I_q} of I_q of an eigenfunction, say Z_k, with the components $\{z_{nk}\}_{k=1,\ldots,N}$ is related to $P \propto P_{N1}$: $P_{I_q}(I_q) = \int \delta_{I_q} P_{N1}(Z_k, e_k, Y) de_k d\tau_k$ with $\delta_{I_q} \equiv \delta(I_q - \Sigma_n |z_{nk}|^{2q})$ and the volume element $d\tau_k$ the same as in Eq. (20). The Y-governed evolution of P_{I_q} can therefore be obtained from Eq. (18) for $P \propto P_{N1}$:

$$\frac{\partial P_{I_q}}{\partial Y} = \frac{\beta^2}{4}(X_1 + X_2) + X_3 \tag{38}$$

where $X_3 = \int \delta_{I_q}(L_E P_{N1}) d\tau_k = 0$ and

$$X_1 = \int \delta_{I_q}(L_{1k} + L_{1k}^*) de_k d^\beta Z_k \tag{39}$$

$$= \frac{4}{\beta}\frac{\partial}{\partial I_q} I_q \int \delta_{I_q} F_1 d\tau_k, \tag{40}$$

$$X_2 = 2\int \delta_{I_q} L_{2k} d\tau_k \tag{41}$$

$$= \frac{8q^2}{\beta^2}\frac{\partial^2}{\partial I_q^2}\int \delta_{I_q} F_2 d\tau_k - \frac{4q(2q+\beta-2)}{\beta^2}\frac{\partial}{\partial I_q}\int \delta_{I_q} F_3 d\tau_k \tag{42}$$

with

$$F_1 = [Q_{nn;k}^{02} + Q_{nn;k}^{02*}],$$

$$F_2 = \sum_{m,n} |z_{mk}|^{2(q-1)} |z_{nk}|^{2(q-1)} z_{nk}^* z_{mk} Q_{mn;k}^{12},$$

and

$$F_3 = \sum_n |z_{nk}|^{2(q-1)} Q_{nn;k}^{12}$$

where L_{1k}, L_{2k}, and $Q_{mn;k}$ are given by Eqs. (19) and (20). Using the approximate form (22) for $Q_{mn;k}^{rs}$, the F's can further be reduced:

$$F_1 \approx q\chi(N-1)/D^2,$$

$$F_2 \approx \frac{\chi}{D^2}\left[\sum_n |z_{nk}|^{2(2q-1)} - \left(\sum_n |z_{nk}|^{2q}\right)^2\right] P_{N1},$$

$$F_3 \approx \frac{\chi}{D^2}\sum_n (|z_{nk}|^{2q-2} - |z_{nk}|^{2q}) P_{N1}, \tag{43}$$

where $\chi=1$ for $\mu < \zeta_k^d$ and $\chi \sim \mu/\zeta_k^d$ for $\mu > \zeta_k^d$.

In general, the fluctuations of different moments (or measures) of the eigenfunction intensity need not be mutually independent. We can therefore define the joint distribution of two measures, say, $h_1(z)$, $h_2(z)$:

$$P_{h_1,h_2}(h_1,h_2) = \int \delta[h_1 - h_1(z)]\delta[h_2 - h_2(z)] P_{N1} de_k d^\beta Z_k. \tag{44}$$

The above definition along with the equality $\Sigma_n |z_{nk}|^{2(2q-1)} = \Sigma_{m,n} |z_{mk}|^{2q} |z_{nk}|^{2(q-1)} - \Sigma_{m,n;m\neq n} |z_{mk}|^{2q} |z_{nk}|^{2(q-1)}$ gives

$$\int \delta_{I_q}\left(\sum_n |z_{nk}|^{2(2q-1)}\right) P_{N1} de_k d^\beta Z_k$$
$$\approx \int I_q I_{q-1} P_{I_{q,q-1}}(I_q, I_{q-1}) dI_{q-1} - \int W P_{I_q,W}(I_q, W) dW$$
$$\approx I_{q-1}^{typ} I_q P_{I_q} - W_q^{typ} P_{I_q} \tag{45}$$

where W_q is a measure of the correlation between the intensities localized at two different basis sates: $W_q = \Sigma_{m,n;m\neq n} |z_{mk}|^{2q} |z_{nk}|^{2(q-1)}$. The second equality in Eq. (45) is obtained from the first by replacing I_{q-1} and W by their typical values; the superscript *typ* over a variable R indicates its typical value: $R^{typ} = \exp\langle \ln R \rangle$.) Using Eqs. (44) and (45), the terms X_1 and X_2 can be rewritten as functions of I_q and P_{I_q}, which in turn leads to

$$\frac{\partial P_{I_q}}{\partial \Lambda_{ip}} \approx 2q \frac{\partial^2}{\partial I_q^2}(I_{q-1}^{typ} I_q - W_q^{typ} - I_q^2) P_{I_q} - \frac{\partial}{\partial I_q}(\alpha I_{q-1}^{typ} - t I_q) P_{I_q} \tag{46}$$

with α, t the same as in Eq. (36) and

$$\Lambda_{ip} = q\chi\Lambda. \tag{47}$$

Note that the above equation along with the definition $\langle I_q \rangle = \int I_q P_{I_q} dI_q$ again leads to Eq. (36).

The behavior of P_{I_q} in different I_q regimes can now be probed by analyzing Eq. (46), using completely localized eigenstates as the initial state. The behavior varies from an exponential decay for the small-I_q regime to log–power law decay for the asymptotic tail regime of I_q:

$$P_{I_q} \simeq \begin{cases} \exp(-\alpha_{i0} I_q), & I_q \lesssim e^{-1} I_q^{typ}, \quad (48) \\ I_q^{-1} \exp\left(-\sum_{n=1}^{M} \alpha_{in} \ln^n (I_q / I_q^{typ})\right), & I_q \gtrsim e^{-1} I_q^{typ}, \quad (49) \end{cases}$$

with $e \approx 2.72$, $\alpha_{i0} \approx t(1-e^{-t\Lambda_{ip}})^{-1}/2q$, and $\alpha_{in} \sim (-1)^n 2 (I_{q-1}^{typ})^n / (2q I_q^{typ})^n$ (valid for $q < N$), and M as a large integer. Note that the alternating $\pm$ signs of terms with increasing powers lead to convergence of the series in the exponent. However, the tail is dominated by increasingly higher powers of the logarithmic term as I_q increases above its typical value. For example, for $e^{-1} I_q^{typ} < I_q \lesssim I_q^{typ}$, $n=1$ dominates the exponent and P_{I_q} behaves as a power law. Similarly, the tail shows a log-normal decay for the regime $I_q^{typ} < I_q \lesssim e I_q^{typ}$.

Equation (46) depends on more than one parameter, namely, Λ_{ip} as well as size-dependent parameters (appearing through t). This suggests absence of single-parameter scaling in IPR distributions. However, as suggested by Eqs. (48) and (49), it seems possible to define a single parameter locally (that is, different single parameters governing different IPR regimes). Further, note that the asymptotic behavior of P_{I_q} is sensitive to the Λ strength and is therefore system specific. This result also agrees with the nonlinear sigma model (NLSM) result obtained for disordered systems [17].

C. Pair function $w(r,r')$

The measure contains important information about the spatial correlations between components of an eigenfunction Z_j at two different basis points of the sample and at an energy e: $w(n,m) = |z_{nj} z_{mj}|^2$ [equivalently $w(r,r') = |z_j(r) z_j(r')|^2$ in a continuous basis, e.g., coordinate space r]. In the localized phase, the asymptotic behavior of $\ln w(r,r')$ at $|r-r'| \to \infty$ determines the rate of exponential decay of the eigenfunction amplitude. It is also useful for many physical applications, e.g., in determination of the form factor of resonance scattering in the complex nuclei or the resonance conductance of the quantum dot with point contacts in the Coulomb blockade regime [15].

The distribution $P_{w,e} = \langle \Sigma_j \delta(w - |z_{nj} z_{mj}|^2) \delta(e - e_j) \rangle$ of the correlation between nth and mth components of an eigenfunction, at an energy e, is related to P_{N1}: $P_w(w,e) = \Sigma_j \int \delta_w \delta_{e,j} P_{N1}(Z_j, e_j, Y) d\tau_j$ with $\delta_w \equiv \delta(w - |z_{nj} z_{mj}|^2)$ and $d\tau_j$ the same as in Eq. (20). Consequently, its rate of change with respect to Y can be determined by Eq. (18),

$$\frac{\partial P_w}{\partial Y} = N L_e P_w + \frac{\beta^2}{4}(A_1 + A_2) \tag{50}$$

where

$$A_1 = \frac{4}{\beta} \sum_j \frac{\partial}{\partial w} w \int \delta_w \delta_{e,j} (Q_{nn;j}^{02} + Q_{mm;j}^{02}) d\tau_j, \tag{51}$$

$$A_2 = \frac{8}{\beta^2} \sum_j \frac{\partial^2}{\partial w^2} w \int \delta_w \delta_{e,j} (F_1 + F_2) de_j d^\beta Z_j$$
$$- \frac{4}{\beta} \sum_j \frac{\partial}{\partial w} \int \delta_w \delta_{e,j} (F_1 + 2\beta^{-1} F_2) d\tau_j \tag{52}$$

with $F_1 = |z_{mj}|^2 Q_{nn;j}^{12} + |z_{nj}|^2 Q_{mm;j}^{12}$ and $F_2 = z_{nj}^* z_{mj} Q_{nm;j}^{12} + z_{nj} z_{mj}^* Q_{mn;j}^{12}$. Equation (50) is derived by first using Eq. (18), followed by repeated partial integration. Note that $\int \delta_w \delta_{e,j} (L_E P_{N1}) de_j dZ_j = N L_e P_w$. Within approximation (20) for the Q's, A_1, A_2 can further be simplified which on substitution in Eq. (50) give the diffusion of P_w in closed form:

$$\frac{\partial P_w}{\partial \Lambda_w} = \frac{\partial^2}{\partial w^2}[w(\Omega_1 - 4w)] P_w - \frac{\partial}{\partial w}(\Omega_2 - bw) + N L_e P_w \tag{53}$$

where $\Omega_1 = |z_{mj}|^2_{typ} + |z_{nj}|^2_{typ} + 2|z_{nj}|^2_{typ} \delta_{nm} = 2(1+\delta_{nm}) u_j^{typ}$, with $u_j^{typ} = \exp(\langle \ln u_j \rangle)$ as the typical local intensity of the jth eigenfunction, $\Omega_2 = (\beta/2)[|z_{mj}|^2_{typ} + |z_{nj}|^2_{typ} + (4/\beta)|z_{nj}|^2_{typ} \delta_{nm}] = (\beta + 2\delta_{nm}) u_j^{typ}$, $\Lambda_w = 2\chi\Lambda$, and $b = \beta N + 2$. The last term on the right of Eq. (53) can be removed by an integration over energy e, leaving us with an evolution equation for the energy-averaged P_w. Note the energy averaging of Eq. (53) for case $n=m$ corresponds to Eq. (46) for $P(I_2)$ [as $\Sigma_n w(n,n) = I_2$].

Exploiting the similarity of the form of energy-averaged Eq. (53) to Eq. (46), the behavior of $P_w(w)$ in different regimes can again be given by Eqs. (48) and (49) after the following replacements (everywhere in the equations): $I_q \to w$, $\alpha_{in} \to \alpha_{wn}$ where $\alpha_{wn} \sim (-1)^n 2^{1-2n} \Omega_2^n (w^{typ})^{-n}$ for $n \gtrsim 1$

and $\alpha_{w0} \approx b(1-e^{-b\Lambda_w})^{-1}/4$. Thus $P_w(w)$ decays exponentially for small w ranges ($w \lesssim w^{typ}/e$): $P_w \simeq \exp(-\alpha_{w0} w)$. It shows a power law behavior for regimes $e^{-1}w^{typ} \gtrsim w \gtrsim w^{typ} P_w \simeq w^{-1} e^{-\alpha_{w1} \ln(\kappa w)}$, a log-normal decay for regimes $w^{typ} \gtrsim w \gtrsim e w^{typ}$. Such a behavior was predicted by nonlinear σ model studies of quasi-1D disordered wires too [3].

Equation (53) can be used to study the behavior of various moments of the distribution of pair-correlation. For example, for average behavior of w, that is, $\langle w \rangle = \int w P_w(w;\Lambda) dw$, Eq. (53) gives its Λ evolution. The evolution equation turns out to be of the same form as Eq. (36) (with $q=2$) with the following replacements: $\langle I_q \rangle \to \langle w \rangle$, $\alpha \to \Omega_2$, and $2\Lambda \to \Lambda_w$ (note $\langle I_{q-1} \rangle = 1$, $t \to b$ for $q=2$). It can be solved to show that $\langle w(\Lambda_w) \rangle = e^{-b\Lambda_w} [\langle w(0) \rangle + (\Omega_2/b)(e^{b\Lambda_w}-1)]$. A choice of $\langle w(0) \rangle = \delta_{ml}$ (corresponding to the localized regime) gives $\langle w(\Lambda_w) \rangle = \delta_{ml} e^{-b\Lambda_w} + (\Omega_2/b)(1-e^{-b\Lambda_w}) \approx (1-\kappa)/N$ which is analogous to the result obtained for disordered systems (by NLSM techniques); see [3].

D. Correlation between eigenfunctions at two different energies

Critical point studies of many systems indicate the presence of multifractal structures among eigenfunctions. The multifractality suggests that the wave function is effectively located in a vanishingly small fraction of the system volume. However, such extremely sparse wave functions can exhibit strong correlations if they belong to neighboring energy levels; the correlations therefore preserve the level repulsion despite the sparsity of the wave function. Thus, for a complete analysis of level statistics and associated physical properties, a knowledge of correlations among eigenfunctions is very important. The correlations are also used in the analysis of many other physical properties, e.g., for the measurement of the linear response of the system, or to determine the fluctuations of matrix elements of some operator in a given basis. This information is useful in studies of the effect of a particular interaction on the statistical properties of the system, e.g., the effect of electron-electron interactions on a single-particle disordered system.

The correlations between components of two eigenfunctions at different energies can be described as $\sigma(n,m,e_k,e_l) = |z_{nk} z_{ml}|^2$ [equivalently, in a continuous basis, $\sigma(r,r',e,e') = |\psi_e(r)\psi_{e'}(r')|^2$]. The distribution P_σ of the correlation $\sigma = |z_{nk} z_{nl}|^2$ between the nth and mth components of the eigenfunctions Z_k and Z_l, respectively, is related to P_{N2}: if $P_\sigma(\sigma,e,\omega) = \Sigma_{k,l} \int \delta_\sigma \delta_{e,k} \delta_{e+\omega,l} P_{N2}(Z_k,Z_l,e_k,e_l,Y) \quad (\Pi_{j=k,l} d\tau_j)$ with $\delta_\sigma \equiv \delta(\sigma - |z_{nk} z_{ml}|^2)$, $d\tau_j$ the same as in Eq. (20), and $\omega = |e_k - e_l|$ as the energy difference between two states. Using Eq. (25) and proceeding as in the case of P_w, the diffusion of $P_\sigma(\sigma)$ can be shown to be described by the equation

$$\frac{\partial P_\sigma}{\partial \Lambda_\sigma} = \left(\frac{\partial^2}{\partial \sigma^2} [\sigma(\tilde{\Omega}_1 - \sigma)] - \frac{\partial}{\partial \sigma} [(\tilde{\Omega}_2 - b\sigma)] + L_e + L_{e+\omega} \right) P_\sigma \tag{54}$$

where $\Lambda_\sigma = 2\chi\Lambda$, $\tilde{\Omega}_1 = (|z_{sl}|^2_{typ} + |z_{rk}|^2_{typ} + 2|z_{rk}|^2_{typ} \delta_{kl}\delta_{rs})$, $\tilde{\Omega}_2 = [\beta(|z_{rk}|^2_{typ} + |z^2_{sltyp}) + 4|z_{rk}|^2_{typ} \delta_{kl}\delta_{rs}]/2$, and $b = \beta N + 2$. Note that Eq. (53) is a special case of the above equation [as $P(w) \equiv P(\sigma)$ for $k=l$].

The energy averaging of Eq. (54) once again leads to an equation similar in form to Eq. (46). Exploiting the analogy, we again get three different regimes for $P_\sigma(\sigma)$: (i) $\simeq \exp(-\alpha_{\sigma 0}\sigma)$ (for $\sigma \lesssim e^{-1}\sigma^{typ}$), (ii) $\simeq \sigma^{-1} \exp[-\alpha_{\sigma 1} \ln(\sigma/\sigma^{typ})]$ (for $e^{-1}\sigma^{typ} \gtrsim \sigma \gtrsim \sigma^{typ}$), and (iii) $\simeq \sigma^{-1} \exp[-\alpha_{\sigma 2} \ln^2(\sigma/\sigma^{typ})]$ (for $e\sigma^{typ}\sigma \gtrsim \sigma^{typ}$) where $\alpha_{\sigma 0} \approx b(1-e^{-b\Lambda_\sigma})^{-1}/4$ and $\alpha_{\sigma n} \sim (-1)^n 2^{1-2n} \tilde{\Omega}_2^n (\sigma^{typ})^{-n}$ for $n \gtrsim 1$.

The Λ dependence of the average behavior of $\langle \sigma \rangle$ can now be derived by multiplying Eq. (54) by σ and then integrating over σ; the equation again turns out to be same as the $q=2$ case of Eq. (36) after the following replacements: $\langle I_q \rangle \to \langle \sigma \rangle$, $\alpha \to \tilde{\Omega}_2$, and $2\Lambda \to \Lambda_\sigma$. Solving the so-obtained evolution equation gives $\langle \sigma(\Lambda_\sigma) \rangle = e^{-b\Lambda_\sigma} [\langle \sigma(0) \rangle + (\tilde{\Omega}_2/b)(e^{b\Lambda_\sigma} - 1)]$. The choice of a localized initial state (e.g., an insulator at $\Lambda_\sigma = 0$) corresponds to $\langle \sigma(0) \rangle = 0$ which gives $\langle \sigma(\Lambda) \rangle \approx (\beta/N)(1-e^{-\beta\Lambda})$.

In this paper, we have considered only two-point correlations. The other correlations, e.g., $\langle z^*_{rk} z_{rl} z_{sk} z^*_{sl} \rangle$ related to linear response of the system, or higher-order ones, e.g., $\langle |z_{rk}|^4 |z_{sk}|^4 \rangle$ related to IPR fluctuations, can similarly be determined using Eq. (25).

E. Local density of states $\rho(e,j)$

The local density of states or the spectral function, defined as $\rho(e,j) = \Sigma_n |U_{jn}|^2 \delta(e-e_n)$, is an important measure of localization. This is because it counts the eigenstates U_n having appreciable overlap with (or equivalently, located close to) the site j. Note that this is distinct from the global density of states $\rho(e)$ which counts all the eigenstates at the energy e irrespective of their location in space. The measure $\rho(e,j)$ is of special interest as it gives information about the decay of a specific unperturbed state into other states due to interaction. The width of the local density of states (LDOS) defines the effective lifetime of the unperturbed basis state. Its distribution is an experimentally accessible quantity related to the position and form of NMR line [3].

The probability density $P_\rho(\rho)$ of $\rho(e,j)$ is related to P_{1N}: $P_\rho(\rho) = \int \delta_\rho P_{1N}(E,Z,Y) dE\, d^\beta Z$ where $\delta_\rho = \delta(\rho - \Sigma_n |z_{jn}|^2 \delta(e-e_n))$ (note that here $Z \equiv \{z_{jn}\}_{n=1,\ldots,N}$). The diffusion of P_ρ due to changing system parameters can again be studied with the help of Eq. (14) for P_{1N}:

$$\frac{\partial P_\rho}{\partial Y} = L_E P_\rho + B + B^* \tag{55}$$

with $B = B^* = \int \delta_\rho L_z P_{1N} d^\beta Z\, dE = \partial^2 F_1/\partial\rho^2 + (\beta/2)\partial F_2/\partial\rho$. Here the second form of B is obtained from the first by a substitution of three terms of L_z [Eq. (15)] in the integral, and a subsequent, partial integration, which gives

$$F_1 = \int \delta_\rho \left(\sum_{m,n;m\neq n} \frac{|z_{jn}|^2 |z_{jm}|^2}{(e_n - e_m)^2} \delta(e-e_n) \times [1 - \delta(e-e_m)] \right) P_{1N} d^\beta Z\, dE, \tag{56}$$

$$F_2 = \int \delta_\rho \left(\sum_{m,n;m\neq n} \frac{|z_{jm}|^2 - |z_{jn}|^2}{(e_n - e_m)^2} \delta(e - e_n) \right) P_{1N} d^\beta Z \, dE. \tag{57}$$

As in the case of the integrals Q and G (see Appendix A), the dominant contribution to the integrals F_1 and F_2 comes from the regions where the exponent term in P_{1N}, that is, $f=(\mu/2)\Sigma_{m<n}|e_n-e_m|^2|U_{jn}|^2|U_{jm}|^2<1$. This can occur under two conditions.

(1) $\mu\langle\sigma\rangle<1$. Here $\sigma=|U_{jn}|^2|U_{jm}|^2$ describes the correlation between two different eigenfunction components in the same basis state. Under this condition, a neighborhood of the order of mean level spacing can contribute to the integral over e variables, i.e., $|e_n-e_m|^2\sim D^2$.

(2) $\mu\langle\sigma\rangle\geq 1$. In this case, $f\gg 1$ only for those regions where $|e_n-e_m|^2\sim D^2/(\mu\langle\sigma\rangle)$. Note, however, in both cases, that almost the entire eigenfunction space can contribute to the integral.

Thus F_1 and F_2 can be approximated as $F_1\approx\chi D^{-2}(\rho(1-\rho)P_\rho)$ and $F_2\approx N\chi D^{-2}(\rho-\langle\rho\rangle)P_\rho$ where $\chi=1$ if $\mu\langle\sigma\rangle\lesssim 1$ and $\chi\sim\mu\langle\sigma\rangle$ for $\mu\langle\sigma\rangle\gtrsim 1$. The approximate forms of F_1 and F_2 can now be used to rewrite B as a function of P_ρ which on substitution in Eq. (55) gives the diffusion equation for P_ρ:

$$\frac{\partial P_\rho}{\partial \Lambda_\rho} = L_e P_\rho + \frac{\partial^2}{\partial \rho^2}[\rho(1-\rho)]P_\rho + \frac{\beta N}{2}\frac{\partial}{\partial \rho}(\rho - \langle\rho\rangle)P_\rho \tag{58}$$

where $\Lambda_\rho=\chi\Lambda$. Note that the above equation is analogous in form to Eqs. (46), (53), and (54) of P_{I_q}, P_w, and P_σ, respectively. This similarity is reflected in both short- as well as long-range behavior of P_ρ: (i) $\simeq\exp(-\alpha_{\rho 0}\rho)$ (for $\rho\lesssim\rho^{typ}$), (ii) $\simeq\rho^{-1}\exp-\alpha_{\rho 1}\ln(\kappa\rho)$ (for $\kappa_1^{-1}\gtrsim\rho\gtrsim\rho^{typ}$), and (iii) $\simeq\rho^{-1}\exp[-\alpha_3\ln^2(\kappa\rho)]$ (for $\sigma\gtrsim\kappa^{-1}$) where $\alpha_{\rho 0}\approx b(1-e^{-b\Lambda_\rho})^{-1}/4$ and $\alpha_{\rho n}\sim(-1)^n 2^{1-2n}(\rho^{typ})^{-n}$.

V. THE PARAMETER Λ

The set of equations (14), (18), and (25) provides a common mathematical formulation for the eigenfunction statistics of various complex systems modeled by Eqs. (1) and (2); here the information about the system enters only through Y. As shown explicitly in [47], the same Y also enters in the common mathematical formulation of the eigenvalue statistics of ensemble (1) [and (2), see [49]]; this is implied by Eqs. (14), (18), and (25) too. However, as discussed in [47,49], the evolution of the nth-order eigenvalue correlations ($n>1$) as a function of Y, is abrupt in the large-N limit; a smooth crossover can only be seen in terms of a rescaled parameter Λ_e where

$$\Lambda_e(e,Y) = \Lambda = \frac{Y - Y_0}{D_\zeta^2} \tag{59}$$

with $D_\zeta(e,Y)=D(\zeta/L)^d$ as the local mean level spacing, $D(e,Y)$ as the mean level spacing of the full spectrum, and ζ as the correlation or localization length for a d-dimensional system of length L ($N=L^d$), at an energy e and parameter Y (with Y_0 as its initial value). Thus Λ_e for various systems, e.g., disordered systems, mixed systems, systems with chiral or particle-hole symmetry, etc., can be calculated by prior knowledge of system parameters (e.g., see [48] for the calculation for Anderson and Brownian ensembles). As Λ_e increases from zero to infinity, the level statistics changes from its initial state (with $Y=Y_0$) to that of the Wigner-Dyson ensemble. For example, let the initial state correspond to the insulator limit of disordered systems or the integrable limit of mixed systems; both limits show Poisson level statistics [3,9,42,53,54]. A variation of system parameters changes Λ from zero, causing diffusion of levels toward the Wigner-Dyson steady state. According to the Λ formulation, the level statistics, for system parameters resulting in finite Λ, is then an intermediate point of the Poisson→Wigner-Dyson transition. The prediction is in agreement with previous works on the two systems [3,20,48,53,54] (note that Wigner-Dyson statistics corresponds to the metallic and chaotic limits of disordered and mixed systems, respectively [23,42]).

For later reference, it is worth reviewing the role of Λ_e, that is, Λ in locating the critical point of level statistics. As both $|Y-Y_0|$ as well as the local mean level density are functions of N, the latter can affect Λ significantly. As a consequence, the size N plays a crucial role in determining the level statistics in the critical regime. For finite systems, the eigenvalue statistics smoothly approaches one of the two end points, namely, $\Lambda\to 0$ or $\Lambda\to\infty$, with increasing system size. The variation of Λ in infinite systems, however, may lead to an abrupt transition of the statistics, with its critical point given by the condition $\Lambda=$size independent (see Ref. [17] for the definition of a critical distribution). The finite, nonzero Λ strength, say $\Lambda_{critical}$, at the critical point results in an eigenvalue statistics different from the two end points. Note, however, that the existence of a critical point or its absence depends on the relative size dependencies of $|Y-Y_0|$ and the local mean level spacing. If the size dependence of D_ζ^2 remains different from that of $|Y-Y_0|$ under all complexity conditions, Λ will never achieve a finite nonzero value in the infinite size limit. As a consequence, such a system will not show a critical behavior of eigenvalue statistics. For example, as discussed in [48] for a d-dimensional Anderson Hamiltonian of linear size L, Λ turns out to be size independent only for $d>2$. The Λ formulation, therefore, indicates the lack of a metal-insulator transition for dimensions $d\leq 2$ which is in agreement with several studies of previous years.

The connection of the eigenvalue fluctuations to those of eigenfunctions suggested Λ as the evolution parameter for the eigenfunctions correlations (of order $n>1$) too. As shown in Sec. III, the evolution parameters $\Lambda_{measure}$ of various eigenfunction fluctuation measures are indeed functions of Λ: $\Lambda_{measure}=f(\Lambda)$. Here $f(\Lambda)\propto\Lambda$ on short length scales and $f(\Lambda)\sim\Lambda e^{-\alpha\Lambda}$ in the tail regime.

The parameter Λ, being a function of the distribution parameters of the matrix elements, is sensitive to changes in the system parameters; this is due to the latter's influence on the uncertainties associated with system interactions. Some examples of such system parameters are disorder, dimensionality, boundary and topological conditions, system size, etc. For example, the presence of disorder randomizes the interactions, with the degree of disorder affecting the distribution

parameters h,b, and consequently Λ. The dependence of Λ on the dimensionality and boundary conditions originates from their influence on the basis connectivity, i.e., degree of sparsity of the matrix, which is reflected in the distribution parameters h,b. For example, for nearest neighbor hopping and hard wall boundary conditions in d dimensions, the matrix element $H_{jk} \neq 0$ only if $j=|k \pm L^{d-1}|$ (with L as the linear size). The variance h_{jk} of the distribution $\rho(H_{jk})$ is therefore finite only for $j=k$ or $|j=k \pm L^{d-1}|$ and is zero for all other j,k. The information about dimensionality in Λ also enters through the local mean level spacing which depends on the correlation volume ζ^d. (See also [48] where the dependence of Λ on system parameters is explained by the example of the Anderson Hamiltonian.)

The system size N is another important parameter which affects the evolution of the measures significantly. As shown in Sec. IV, it appears independently as well as through $\Lambda_{measure}$ in the evolution equations, which suggests a two-parametric dependence, namely, $\Lambda_{measure}$ and N (separately), of these measures. As a consequence, even at the critical point of level statistics, the eigenfunction statistics remains sensitive to size N. This in turn results in a multifractal behavior of the eigenfunctions at the critical point of any complex system, modeled by Eqs. (1) and (2). The scaling exponents at the critical point, referred to as critical exponents or multifractal dimensions, depend on the system parameters. In finite size systems, changing system parameters can change Λ (and therefore $\Lambda_{measure}$) continuously between 0 and ∞, which may lead to intermediate stages of varying degrees of multifractality. However, the physically interesting cases usually correspond to infinite sizes where Λ takes only three possible values, namely, $\Lambda=0,\infty,\Lambda_{critical}$; for these cases therefore only one multifractal stage, that is, at the critical point corresponding to $\Lambda_{critical}$, can exist. As $\Lambda_{critical}$ is sensitive to system specifics, the critical (multifractal) exponents can vary from system to system. Note that, as already mentioned above, the occurrence of the critical point and, therefore, a multifractal behavior of eigenstates is not a necessary feature of all infinite size complex systems.

The Λ-governed diffusion equations, derived in Sec. IV, are valid for arbitrary initial conditions at $\Lambda=0$ (which implies $\Lambda_x=0$) and their solutions $P_x(X,\Lambda_x|X_0,0)$ describe the probability of the measure, say X, at Λ_x for a given initial state of $X=X_0$. Thus P is subjected to an initial constraint $\lim \Lambda \to 0 P(X,\Lambda_x|X_0,0)=\delta(X-X_0)$. By integration of the solution over the distribution of initial values $P_0(X_0,0)$, one can recover $P(X,\Lambda_x)$, that is, the distribution of measure X for a system with complexity parameter strength Λ:

$$P(X,\Lambda_x)=\int P(X,\Lambda_x|X_0,0)P_0(X_0,0)dX_0. \tag{60}$$

Equation (60) implies that the statistics evolved in "time" Λ_x is sensitive to the collective behavior of system parameters contributing to Λ_x and the initial distribution only. The latter can always be chosen same for the systems operating in the matrix spaces of similar type, e.g., same symmetry conditions (the initial values of their Y parameters need not be equal). Thus if both A and B operate in matrix spaces of the same type, their behavior at the system parameter strengths which lead to $\Lambda_{x,A}=\Lambda_{x,B}=t$ will also be the same (although they may show different behavior between $0<\Lambda_{x,A},\Lambda_{x,B}<t$). This implies a great deal of universality among systems of widely different origins of complexity. For example, consider the cases of a three-dimensional disordered system, say A, and a clean, closed quantum dot, say B. In the first case, $\Lambda_{x,A}=\Lambda_{x,disorder}$ is a function of disorder, hopping strength, dimensionality, boundary condition, etc. In the case of a dot, $\Lambda_{x,B}=\Lambda_{x,dot}$ is a function of shape as well as size. It is well known that, in the strong disorder limit and for a circular shape, respectively, both systems show localized wave function dynamics and the same statistical behavior of the eigenfunctions and eigenvalues of the Hamiltonians. Reducing the degree of disorder or change of shape of the dot from circle to stadium type results in a transition from localized to delocalized dynamics of the wave functions. The statistics in the intermediate stages during the transition for each case is governed by the respective Λ_x strengths. If, however, $\Lambda_{x,dot}=\Lambda_{x,disorder}$ at some shape parameter and disorder strength, respectively, our analysis predicts the same statistical behavior for both systems. The implication can also be extended to classical systems, e.g., stock market fluctuations, which are analyzed by statistical studies of the correlation matrix of stocks [5]. [Note that the correlation matrices of classical systems are, in general, non-Hermitian; however, as shown in [40], the Λ formulation remains valid for the non-Hermitian version of Eq. (1)]. Here a very weak interaction among certain stocks due to various socioeconomic conditions results in a localized dynamics of the eigenfunctions. The changing conditions may lead to a more homogenized interaction of some of the stocks, thus introducing a transition from localized to delocalized wave dynamics. In this case, Λ is a function of the socioeconomic parameters (SEPs). However if $\Lambda_{x,stock}=\Lambda_{x,dot}$ for some combinations of SEPs and dot parameters, respectively, the spectral and strength fluctuations in correlation matrix of the stock market and the Hamiltonian of quantum dot will show the same behavior. Note that the analogy of statistical behavior of the eigenvalues and eigenfunctions among the three systems, mentioned above, has already been numerically verified in the delocalized wave limit $\Lambda \to \infty$ [1,3,5].

The above universality makes the Λ formulation useful as it can be exploited to obtain the statistics of a complex system if the same information is available about another system (under the same symmetry conditions) by another method. For example, for an Anderson-type disordered Hamiltonian, the distributions of many measures are known by nonlinear σ model techniques. The formulations can then be used for complex systems, e.g., stock markets undergoing a transition from localized to delocalized wave dynamics; one just needs to replace Λ (Anderson) by that of the system.

The formulation can also be used to search for the system conditions leading to a critical state or multifractal wave functions of various complex systems. For example, the suggested modeling of mixed systems by Eq. (1) would imply the possible existence of a critical point of level statistics in the systems and multifractal eigenstates. The intuition suggests that the occurrence of such a point may correspond to breaking of the last KAM curve, thus allowing classical dif-

fusion or delocalization of the dynamics above the critical point and localization below it; however it needs to be further explored. The critical Λ can then be given by the critical value of the system parameter leading to the last KAM curve breaking.

VI. NUMERICAL ANALYSIS

For numerical analysis, we choose three different ensembles (for both cases $\beta=1,2$); the choice is dictated by the reasons (i) that the ensembles are prototype models of many physical systems related to different areas [1,3,8,9,35] and (ii) that a comparative study of the eigenvalue fluctuations of these systems has already been carried out, with their Λ parameters and other results given in [48].

(a) Critical Anderson ensemble (AE): Time-reversal case AE_t. We analyze cubic ($d=3$) Anderson lattice of linear size L ($N=L^d$) with a Gaussian site disorder (of variance $W^2/12$, $W=4.05$, and mean zero), the same for each site, an isotropic random hopping between nearest neighbors with hard wall boundary conditions [3,48]. The ensemble density in this case can be described by Eq. (1) with $h_{kk}=W^2/12$, $h_{kl}=f(kl)/12$, $b_{kl}=0$ where $f(kl)=1$ for $\{k,l\}$ pairs representing hopping, $f(kl)\to 0$ for all $\{k,l\}$ values corresponding to disconnected sites. A substitution of the above values in Eq. (8) gives Y which subsequently gives Λ by Eq. (59) [see Eq. 19 of [48]]:

$$\Lambda_a(E,Y)=|\alpha-\alpha_0|F^2\zeta^{2d}L^{-d}\gamma^{-1} \tag{61}$$

with $\alpha-\alpha_0=1.36$ and $F(E)=0.26e^{-E^2/5}$ (see Sec. V of [48]). [Note, for later reference, that $F(E)$ is the mean level density: $F(E)=(ND)^{-1}$.]

(b) Critical Anderson ensemble: Broken time-reversal case AE_{nt}. We analyze a cubic ($d=3$) Anderson lattice of linear size L ($N=L^d$) with a Gaussian site disorder (of variance $W^2/12$, $W=21.3$, and mean zero), the same for each site, an isotropic nonrandom hopping $t=1$ between nearest neighbors with periodic boundary conditions [3,48]. The time-reversal symmetry is broken by applying an Aharnov-Bohm flux ϕ which gives rise to a nearest neighbor hopping $H_{kl}=\exp(i\phi)$ for all k,l values related to the nearest-neighbor pairs [58]. The flux ϕ is chosen to be nonrandom in nature, that is, $\langle\cos^2(\phi)\rangle=W_1=0$, $\langle\sin^2(\phi)\rangle=W_2=0$ and $\langle\cos(\phi)\rangle=t_1=1$, $\langle\sin(\phi)\rangle=t_2=1$. The ensemble density in this case can again be described by Eq. (1) with $h_{kk}=W^2/12$, $b_{kk}=0$, $h_{kl;s}=W_s=0$, $b_{kl;s}=t_sf(kl;s)$ where $f(kl;s)=1$ for $\{k,l\}$ pairs representing hopping, $f(kl;s)\to 0$ for all $\{k,l\}$ values corresponding to disconnected sites. The Λ for this case is still given by Eq. (61) (except for a factor β^{-1}), however now $\alpha-\alpha_0=5.43$, $F(E)=0.016e^{-E^2/400}$ (see Sec. V of Ref. [48]).

(c) Critical power law random banded matrix ensemble (critical PRBM or PE). As mentioned in Sec. II, the PRBM ensemble was introduced as a possible model for the level statistics of the Anderson Hamiltonian [34]. It is defined as the ensemble of random Hermitian matrices with matrix elements H_{ij} as independently distributed Gaussian variables with zero mean, i.e., $\langle H_{ij}\rangle=0$ and a power law decay of the variances away from the diagonal [3,10,35], $\langle|H_{ij;s}|^2\rangle=a(|i-j|)$ with the function $a(r)$ decaying as r increases.

The PRBM ensemble with the specific choice $\langle|H_{ij;s}|^2\rangle=G_{ij}^{-1}[1+(|i-j|/b)^2]^{-1}$, $G_{ij}=\beta(2-\delta_{ij})$, and $G_{ij}=1/2$ (referred to as the critical PRBM or PE in this paper) leads to a critical behavior of eigenfunction and eigenvalue statistics at arbitrary values of the parameter b and is believed to show all the key features of the Anderson critical point, including multifractality of eigenfunctions and the fractional spectral compressibility [3,34]. The ensemble density in this case corresponds to Eq. (1) with $b_{kl}=0$, and, $h_{kl;s}=G_{kl}^{-1}[1+(|k-l|/p)^2]^{-1}$. The corresponding Λ can be shown to be given by (see Sec. VI of [48]),

$$\Lambda_p(p,E)=\alpha_p^{-1}f(p)F^2(E)\zeta^2N^{-1}, \tag{62}$$

where $\alpha_p=2N(N+2-\beta)$, $f(p)=\Sigma_{r=1}^{N}(N-r)\ln|1+(p/r)^2|$.

(d) Critical Brownian ensemble. A Brownian ensemble can be described as a nonstationary state of the matrix elements undergoing a crossover due to a random perturbation of a stationary ensemble by another one [8,31,48]. For example, in the case of Hermitian operators, a Brownian ensemble H can be given as $H=\sqrt{f}(H_0+\lambda V)$ [with $f=(1-\lambda^2)^{-1}$]; here V is a random perturbation of strength λ, taken from a stationary ensemble [37], e.g., the Wigner-Dyson ensemble, and applied to an initial stationary state H_0 (see also [8,47,48]). Here we consider a specific class of BEs, namely, those appearing during a transition from the Poisson $\to$ Wigner-Dyson ensemble, caused by a perturbation of the former by the latter (that is, taking H_0 and V as the Poisson and Wigner-Dyson ensembles, respectively). In the above two cases this transition also results in a change of localized eigenstates to delocalized ones. The BEs related to the Poisson $\to$ Wigner-Dyson transition can be described by an $N\times N$ ensemble H represented by Eq. (1) with mean $\langle H_{kl}\rangle=b_{kl}=0$, variance $\langle H_{kk;s}^2\rangle=h_{kk;s}=(2\gamma)^{-1}$ and $\langle H_{kl;s}^2\rangle=h_{kl;s}=[4\gamma(1+\mu)]^{-1}$ for $k\neq l$. with $(1+\mu)=(\lambda^2 f)^{-1}$; here $H=H_0$ for $\lambda\to 0$ or $\mu\to\infty$. As mentioned in Sec. II, the ensemble density in this case has the same form as for the Rosenzweig-Porter (RP) ensemble [32]; it can also describe an ensemble of Anderson Hamiltonians with very long-range, isotropic, random hopping. Further, as discussed in [48], the special case $\mu=cN^2$ corresponds to the critical BEs; their mean level density is given as $F(E)=(\pi)^{-1/2}e^{-E^2}$ and

$$\Lambda_b(E)=(1/4c\pi\gamma)e^{-E^2}. \tag{63}$$

Our aim is to show that the behavior of an eigenfunction fluctuation measure of AE, BE, and PE is analogous at system parameters that lead to the same $\Lambda_{measure}$ value for all three cases. Using the latter as a condition, we can obtain the desired system parameters in each case (that is, p for the PE and c for the BE for a given AE). As Λ for the three cases is energy dependent, the fluctuation measures should be compared at precisely a given value of energy. For numerical analysis, however, one needs to consider averages over an energy range ΔE which should be sufficiently large in order to improve the statistics. On the other hand, choice of a very large ΔE will lead to mixing of different statistics (in the range $\Delta\Lambda\propto\delta E$). As a consequence, one needs to consider an optimized range of ΔE. In our simulations, we analyze large

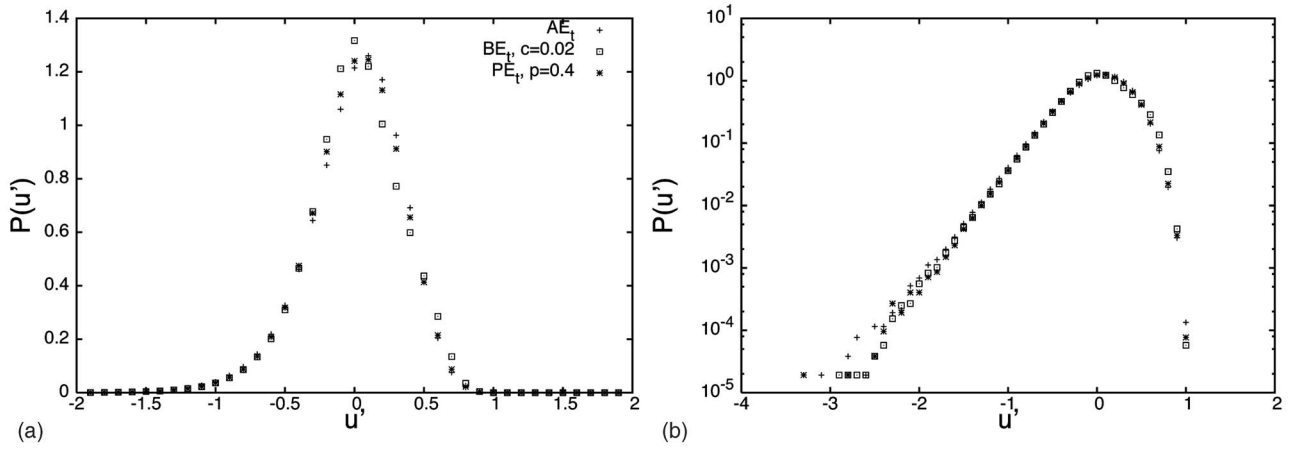

FIG. 1. Distribution $P_u(u')$ with $u'=[\ln u-\langle u\rangle]/\langle \ln^2 u\rangle$ of the local intensity of an eigenfunction near band center for AE_t (cubic lattice of linear size $L=13$, with hard wall boundary conditions, random hopping, and time-reversal symmetry) and its BE and PE analogs. (a) and (b) show short- and long-range behavior of the distribution, respectively. The analogs are obtained by the relation $I_{2,a}^{typ}/F_a(0)=I_{2,b}^{typ}/F_b(0)=I_{2,p}^{typ}/F_p(0)$. For $N=2197$, we find $I_{2,a}^{typ}=0.018$, $I_{2,b=0.02}^{typ}=$, $I_{2,p=0.4}^{typ}=0.02$, and, $F_a(0)=0.26$, $F_{b=0.02}(0)=(\pi)^{-1/2}$, $F_{p=0.4}(0)=0.39$.

ensembles of about 1400 matrices of size $N=2197$. We choose ΔE to be about 10% of the bandwidth at the band center $E=0$ which gives approximately 3×10^5 levels for each case. As the chosen ΔE corresponds to a 1% variation of the density of states, it avoids mixing of different statistics.

As discussed in the previous section, the eigenfunction fluctuations are influenced by both Λ as well as the system size N. To compare the $\Lambda_{measure}$ dependence of a fluctuation measure (of eigenfunctions), therefore, the same system size should be taken for all systems under consideration. As examples, here we consider distributions of three measures, namely, the local eigenfunction intensity $P_u(u)$, the inverse participation ratio $P_I(I_2)$, and the pair correlation function $P_w(w)$ for the three systems under time-reversal symmetry ($\beta=1$), i.e., AE_t, BE_t, PE_t. As, for P_u,

$$\Lambda_u=\frac{\mu(Y-Y_0)\zeta^{2d}}{N^2D^2}\approx\left(\frac{F}{I_2^{typ}}\right)^2, \tag{64}$$

[with $\zeta^d\approx(I_2^{typ})^{-1}$], the BE and PE analogs for the intensity distribution of AE_t can be obtained by the condition $I_{2,a}^{typ}/F_a=I_{2,b}^{typ}/F_b=I_{2,p}^{typ}/F_p$. This requires prior information about I_2^{typ}. Our numerical study for various sizes of the three systems shows that, for each case, $I_2^{typ}\approx\tilde{I}N^{-D_2}$ with $\tilde{I}$ and D_2 system dependent. The numerical information about I_2^{typ} and F can now be used to obtain the parameters p and c for the PE and BE analogs of AE_t for the P_u case (i.e., the PE and BE with the ratio I_2^{typ}/F the same as for the AE); we find $p=0.4$, $c=0.02$. Figure 1 shows the distribution $P_u(u')$, $u'=[\ln u-\langle\ln u\rangle]/\langle\ln^2 u\rangle$, for the BE_t case ($c=0.02$) and PE_t case ($p=0.4$) along with the AE_t case; the close agreement among the three cases confirms our theoretical prediction. This is also confirmed by the comparison of $P_I(\ln I_2)$ and $P_w(\ln w)$ for the three systems, shown in Figs. 2 and 3, respectively. Here again the parameters p and c for the PE and BE analogs for both measures are obtained by the relation $\Lambda_{I,a}=\Lambda_{I,b}=\Lambda_{I,p}$ (similarly for w).

The above numerical analysis is repeated also for the case of an AE in a magnetic field and its BE and PE analogs; the results for the three measures, shown in Figs. 4–6, further support our claim: the eigenfunction fluctuations of different complex systems show the same behavior if their complexity parameters and sizes are equal. It is worth recalling that the behavior of the eigenvalue fluctuations is governed only by the related complexity parameter (that is, there is no independent influence of size). The details of analytical and numerical evidence about the eigenvalue statistics are already published in [47–49]. However, for the sake of completeness and to convince the reader, we include here the numerical analysis of an eigenvalue fluctuation measure, namely, the nearest neighbor spacing distribution $P(S)$ for the three systems (for both $\beta=1,2$ cases) at parametric values leading to $\Lambda_{e,a}=\Lambda_{e,b}=\Lambda_{e,c}$ (where $\Lambda_e=\Lambda$); the plots shown in Figs. 7 and 8 reconfirm the claim about eigenvalue statistics.

VII. CONCLUSION

Finally, we summarize our main results. Our analysis of the eigenfunction correlations of complex systems indicates a two-parameter dependence, namely, the complexity parameter Λ and system size N, of the distributions of eigenfunction components. The independent appearance of the size parameter (besides through Λ) seems to suggest a lack of finite size scaling in eigenfunction distributions and an absence of their critical limit. This is in contrast with the behavior of the eigenvalue distribution which shows a single-parametric scaling as well as a critical limit if the condition $\lim N\to\infty$, $\Lambda=$finite is satisfied by the system. Note that the above implies the size dependence of the eigenfunction correlations at the critical point of level statistics too.

We have also studied the distribution of a few important measures of eigenfunction correlations, e.g., local density of states, pair correlation function, etc. We find that the form of complexity parameter governing an eigenfunction fluctuation measure is sensitive to its nature (e.g., Λ_u for the local inten-

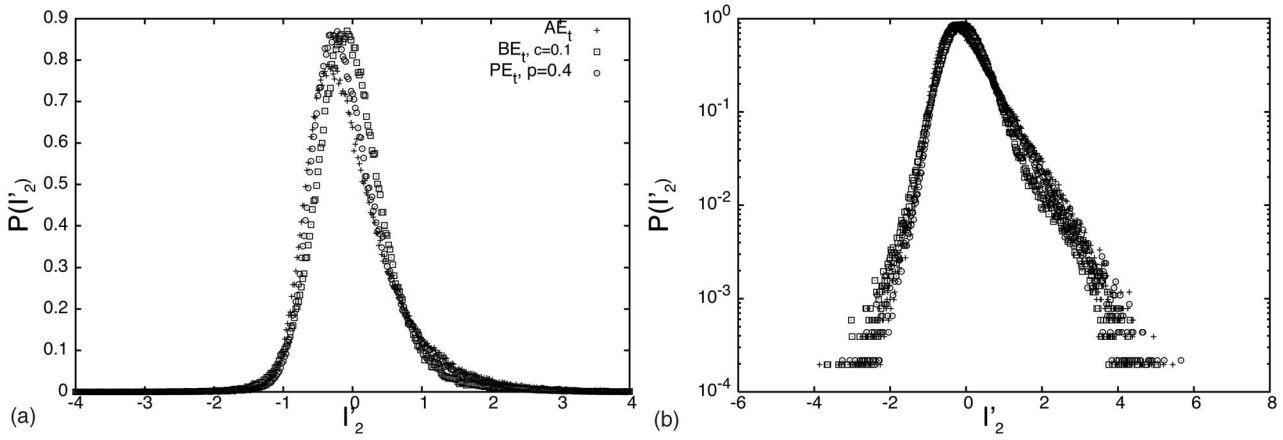

FIG. 2. Distribution $P(I_2')$ of the rescaled inverse participation ratio $I_2'=\ln(I_2/I_2^{typ})$ for AE_t [as in Fig. 1(a)] and its BE and PE analogs: (a) short-range behavior (lin-lin plot); (b) tail behavior (lin-log plot). Here the BE and PE analogs are obtained by the relation $\Lambda_{I,a}=\Lambda_{I,b}=\Lambda_{I,p}$. This gives a BE analog of AE different from that in Fig. 1 although the PE analog remains unaffected; the reason lies in the almost similar mean level density behavior near band center for the AE and PE cases.

sity distribution, Λ_I for the inverse participation ratio distribution, etc.). This is again different from the eigenvalue fluctuations (except for the level density) which are all governed by the same complexity parameter, namely, $\Lambda_e=\Lambda$. Our analysis indicates a log-normal behavior of the asymptotic tails of the distributions at finite Λ strength. In the context of disordered systems, a similar behavior was predicted by other studies using different techniques, e.g., the Berezinski and Abrikosov-Ryzkhin technique (for one dimension) and the nonlinear σ model (for higher dimensions) [3]. However, the complexity parameter formulation suggests the existence of such a tail behavior and multifractal eigenfunctions for almost any complex system, irrespective of the origin of complexity, if the parameter $N\Lambda_{measure}$ is finite. A recent numerical study of the eigenfunction of the correlation matrix of stock prices confirms the suggestion in the case of the stock market [5,27]. As finite Λ corresponds to the critical point condition in infinite size systems, a log-normal tail behavior seems to be associated with the existence of a critical point (and vice versa). The above study can thus be used to search for and predict the critical stages of other complex systems, e.g., the stock market, brain, etc.

In this paper, we have considered cases modeled by generalized Gaussian ensembles with uncorrelated matrix elements as well as a wide range of non-Gaussian ensembles with correlated matrix elements. The latter are suitable models, for example, for disordered systems with varying degrees of particle-particle interactions. In the context of disordered systems, therefore, we expect the same statistical behavior of a measure for both cases, namely, with or without particle interactions, if the strengths of their parameters $\Lambda_{measure}$ are equal. This suggests the sensitivity of the statistical behavior of a disordered system to the degree of its complexity only (measured by the complexity parameter), irrespective of the origin. The statement is expected to be valid for correlated and uncorrelated cases of other complex systems too. This in

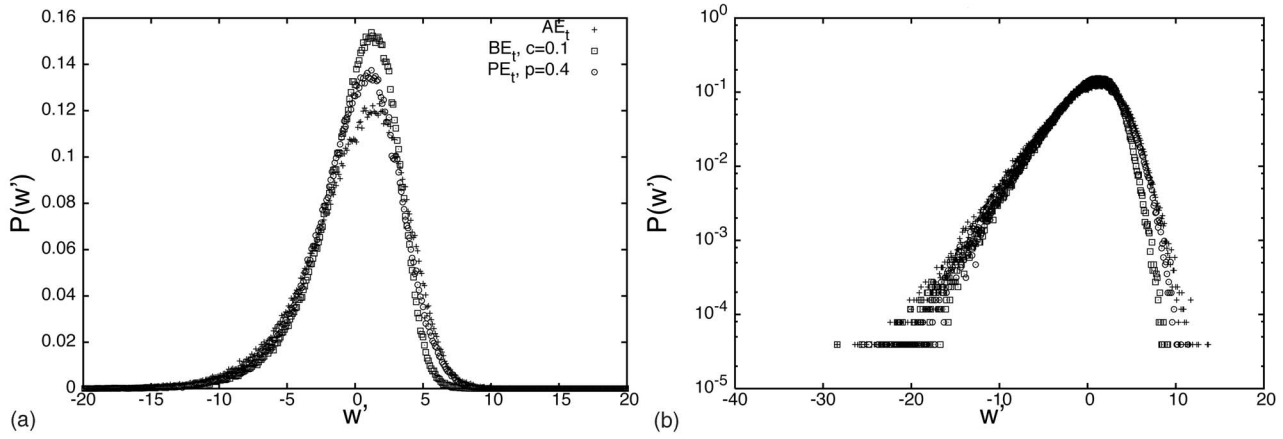

FIG. 3. Distribution $P(w')$ of the spatial correlation $w'=\ln w=\ln|z_{1n}z_{Nn}|^2$ between two points belonging to opposite ends of the sample: (a) short-range behavior (lin-lin plot); (b) tail behavior (lin-log plot). The cases compared here are AE_t [as in Fig. 1(a)] and its BE and PE analogs (obtained by the relation $\Lambda_{w,a}=\Lambda_{w,b}=\Lambda_{w,p}$). Again the BE analog of AE and PE in this case turns out to be different from that in Fig. 1 but the same as in Fig. 2.

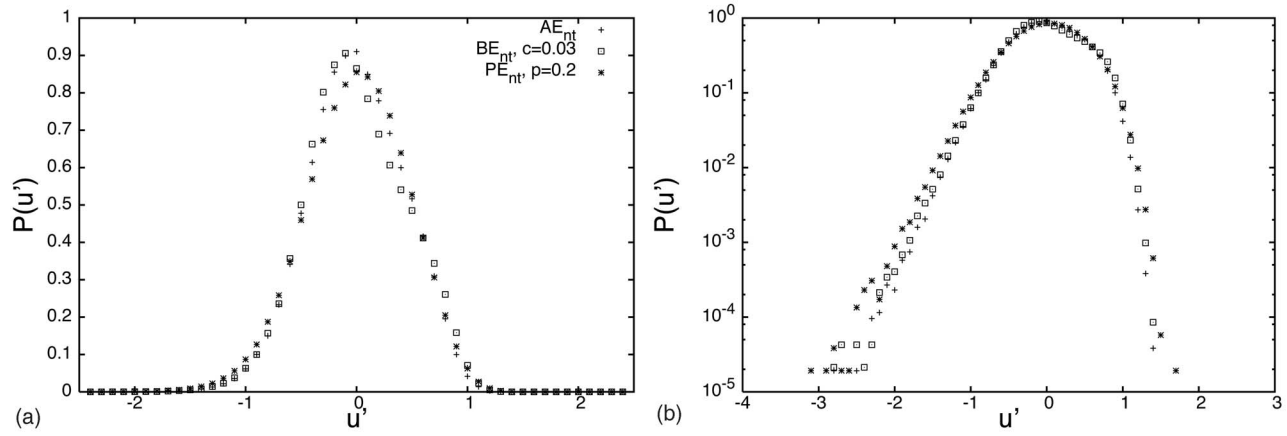

FIG. 4. Local intensity distribution for AE_{nt} (cubic lattice of linear size $L=13$, with periodic boundary conditions, nonrandom hopping, and no time-reversal symmetry) and its BE and PE analogs. In this case, $I_{2,a}^{typ}=0.013$, $I_{2,b=0.03}^{typ}=$, $I_{2,p=0.4}^{typ}=0.000\,45$, and, $F_a(0)=0.016$, $F_{b=0.03}(0)=(\pi)^{-1/2}$, $F_{p=0.4}(0)=0.4$ for $N=2197$. Other details are the same as in Fig. 1.

turn would indicate the existence of an infinite family of universality classes, parametrized by Λ, of statistical behavior among complex systems.

APPENDIX A: CALCULATION OF INTEGRALS $Q_{mn;k}^{rs}$ AND G_r

The integral $Q_{mn;k}^{rs}$ defined by Eq. (20) can be rewritten in terms of $\rho(H)$ as

$$Q_{mn;k}^{rs}=\sum_{j;j\neq k}\int\frac{(U_{nj}U_{mj}^{*})^{r}}{(\lambda_k-\lambda_j)^{s}}\tilde{f}_k\rho(H,Y)dH. \tag{A1}$$

To express Q in terms of P_{N1}, it is necessary to write $\rho(H)$ in eigenvalue-eigenvector space, i.e., $\{\lambda,U\}$ space. The steps can briefly be given as follows. The solution of Eq. (6) for arbitrary initial conditions, say H_0 at $Y=Y_0$, can be given as $\rho(H,Y|H_0,Y_0)\propto\exp[-(\alpha/2)\mathrm{Tr}(H-\eta H_0)^2]$ with $\alpha=\gamma(1-\eta^2)^{-1}$ and $\eta=e^{-\gamma Y}$. Without loss of generality, the basis space for H can be chosen as the eigenvector space of H_0; the initial ensemble H_0 in this basis consists of diagonal matrices. For simplification, consider the initial distribution given by $\rho(H_0)\propto\exp(-\Sigma_{j=1}^{N}H_{0;jj}^2)$. Using the eigenvalue equation $UH=\Lambda U$, $\rho(H,Y|H_0,Y_0)$ can be transformed from matrix space to eigenvalue-eigenvector space $\{\lambda,U\}$ which, followed by an integration over ensemble H_0, gives

$$\rho(H,Y)\propto\prod_{k,l;k<l}^{N}|\lambda_k-\lambda_l|^{\beta}\times\exp\left(-(1/2)\sum_{j=1}^{N}\lambda_j^2-(\mu/2)\sum_{k<l}|\lambda_k-\lambda_l|^2U_{jk}^2U_{jl}^2\right) \tag{A2}$$

where $\mu=(e^{2\eta(Y-Y_0)}-1)^{-1}$.

Substitution of Eq. (A2) for ρ in Eq. (A1) and using $dH=\Pi_j\Pi_{k<l}|\lambda_k-\lambda_l|^{\beta}d\lambda_j dU_j$ gives Q as a function of $\{U,\Lambda\}$ variables. As Eq. (A2) indicates, the behavior of $Q_{mn;k}^{rs}$ is significantly influenced by the term $R\equiv\mu|\lambda_k-\lambda_j|^2\Sigma_{n=1}^{N}|U_{nk}|^2|U_{nj}|^2$

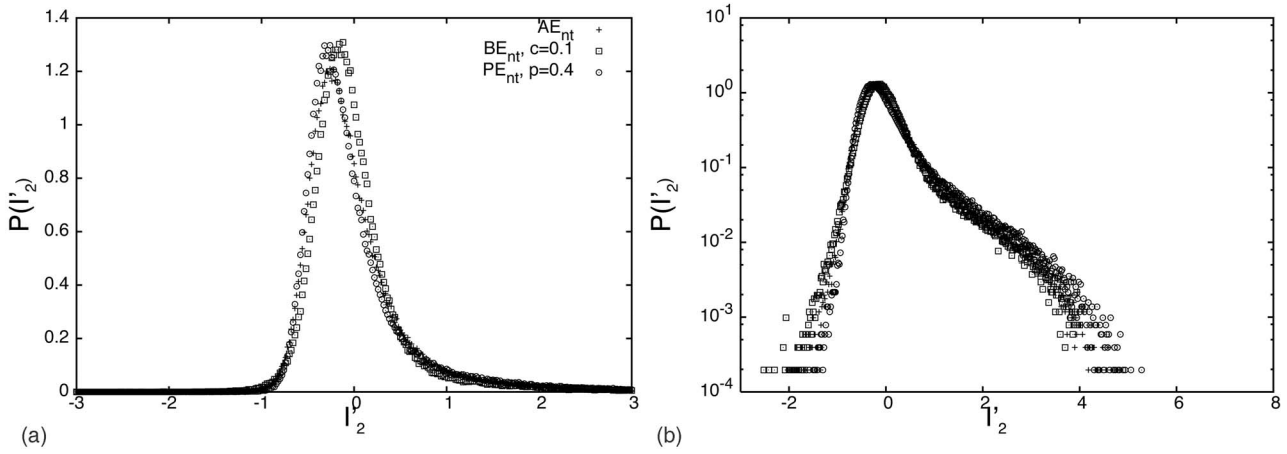

FIG. 5. Distribution $P(I_2')$ of the rescaled inverse participation ratio I_2' for AE_{nt} [as in Fig. 1(b)] and its BE and PE analogs. The other details are as in Fig. 2. Note the BE analog of AE and PE in this case is different from that in Fig. 4.

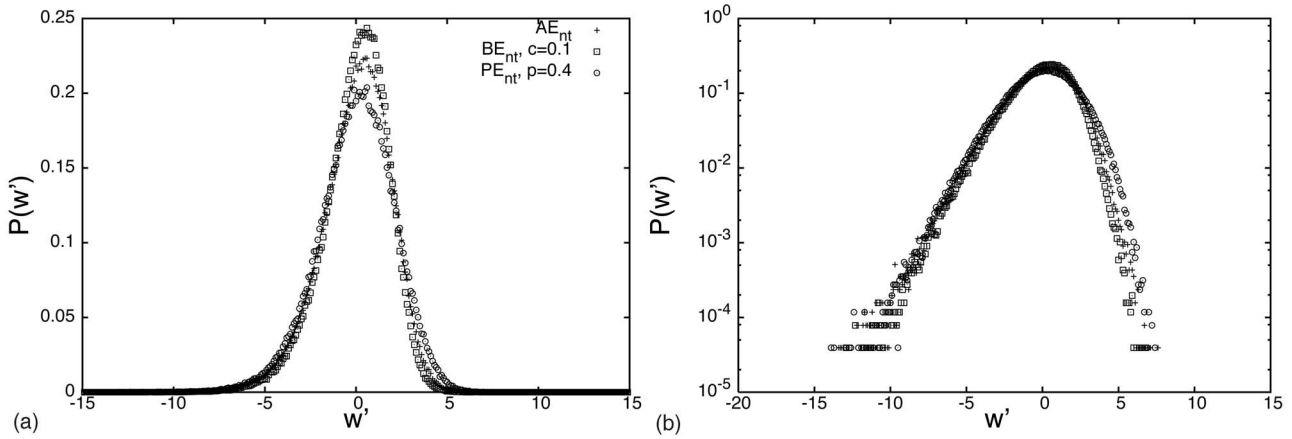

FIG. 6. Distribution $P(w')$ of the spatial correlation w' for AE_{nt} (as in Fig. 4) and its BE and PE analogs; other details are the same as in Fig. 3. Here the BE and PE analogs are obtained by the relation $\Lambda_{w,a}=\Lambda_{w,b}=\Lambda_{w,p}$. The BE analog of AE and PE in this case is different from that in Fig. 4 but the same as that in Fig. 5.

present in the exponent of ρ. Consequently, for a given Y, the dominant contribution to the integrals over the variables U_j and λ_j in Eq. (A1) comes from those regions which lead to $R\to 0$. Also note that the eigenvalue-eigenfunction correlations appear in ρ only through R. The limit $R\to 0$ therefore allows a mutually independent integration over λ_j and U_{nj} variables. As the typical local intensity $|U_{nk}|^2_{typical}\sim \zeta_k^d$ with ζ_k as the localization length of the eigenfunction U_k (d as system dimension), this implies $R\sim \mu\zeta_k^d|\lambda_k-\lambda_j|^2$. Consequently, the regions of variable λ_j and U_{nj} which contribute to the integral depend on mutual competition between μ and ζ_k^d.

(i) For $\mu<\zeta_k^d$, almost the entire region of U_{nj} can contribute to the integral (since $0<|U_{nj}|^2<1$). However, only a small neighborhood of the order of the local mean level spacing, i.e., $|\lambda_k-\lambda_j|\simeq D_k$ around λ_k, contributes to λ_j integration. Here D_k is the local mean level spacing at eigenvalue λ_k. As a consequence, an approximation of the repulsion term $|\lambda_k-\lambda_j|\approx D_k$ along with the relation $\Sigma_{k=1}^N U_{nk}U_{mk}^*=\delta_{mn}$ (due to the unitary nature of U) gives

$$\sum_{j=1;\neq k}^{N}\frac{(U_{nj}U_{mj}^*)^r}{(\lambda_k-\lambda_j)^s}=\frac{(\delta_{mn}-U_{nk}U_{mk}^*)^r}{(N-1)^{r-1}D_k^s}. \tag{A3}$$

Here $r=0,1$ only, $\chi=1$, and D_k is the local mean level spacing at eigenvalue λ_k.

(ii) For $\mu>\zeta_k^d$, the significant contribution comes from the regions of λ_j where $|\lambda_j-\lambda_k|\sim D_k(\zeta_k^d/\mu)^{1/2}$. Here again, as a typical $|U_{nj}|^2\sim\zeta^{-d}<1$, the entire region of U_j can contribute to the integral. Consequently one can approximate

$$\sum_{j=1;\neq k}^{N}\frac{(U_{nj}U_{mj}^*)^r}{(\lambda_k-\lambda_j)^s}=\left(\frac{\mu}{\zeta_k^d}\right)^{s/2}\frac{[\delta_{mn}-U_{nk}U_{mk}^*]^r}{(N-1)^{r-1}D_k^s}. \tag{A4}$$

[One may also consider the contribution from regions where $|U_{nj}|^2<(\mu|U_{nk}|^2D_k^2)^{-1}$; however it is weaker than the above.]

By substituting the approximations (A3) and (A4) in Eq. (A1), $Q_{mn;k}^{rs}$ can be written as (for $r=0,1$ only)

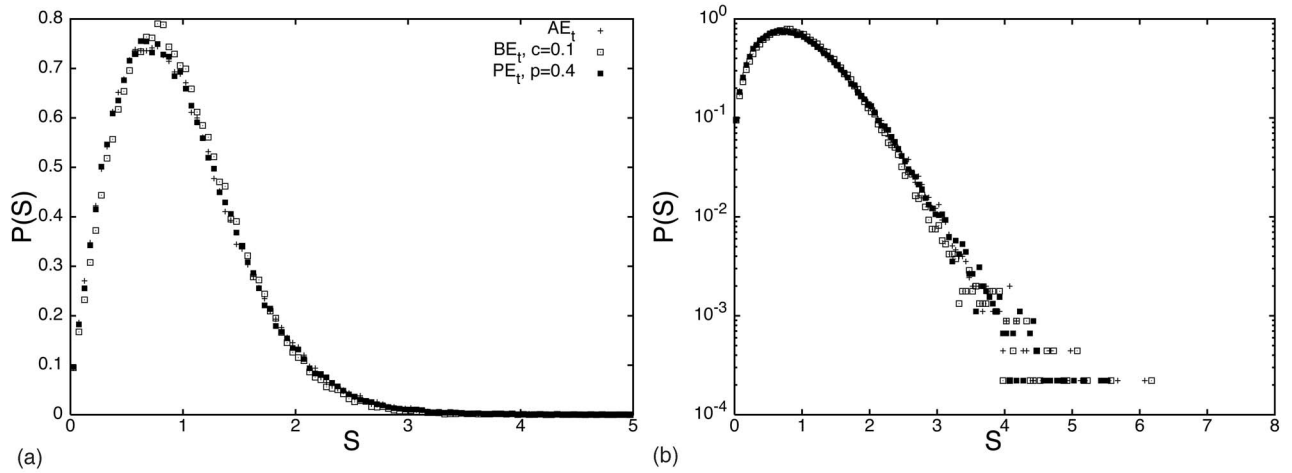

FIG. 7. Distribution $P(S)$ of the nearest-neighbor spacing distribution S of the eigenvalues, with (a) and (b) showing short- and long-range behavior, respectively, for AE_t (as in Fig. 1) and its BE and PE analogs. Here the BE and PE analogs are obtained by the relation $\Lambda_{e,a}=\Lambda_{e,b}=\Lambda_{e,p}$. Note that the BE analog of AE and PE in this case is different from that in Fig. 1 but the same as in Figs. 2 and 3.

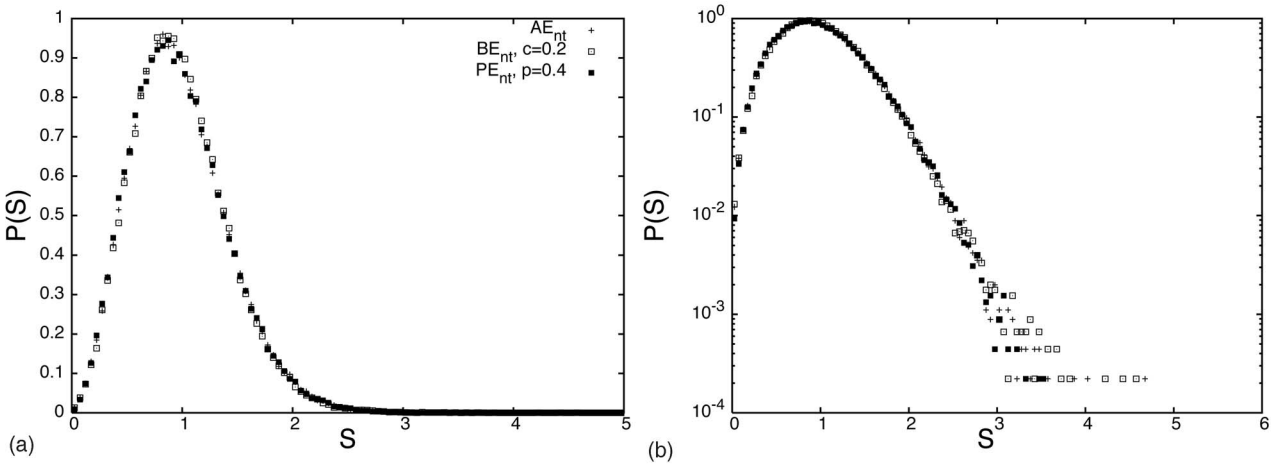

FIG. 8. Distribution $P(S)$ of the nearest-neighbor spacing distribution S for the case AE_{nt} (as in Fig. 4) and its BE and PE analogs (other details are the same as in Fig. 7).

$$Q^{rs}_{mn;k} \approx \chi^{s/2} \frac{(\delta_{mn} - z^*_{mk} z_{nk})^r}{(N-1)^{r-1} D^s} P_{N1}(Z_k, e_k) \tag{A5}$$

where $\chi=1$ for $\mu < \zeta^d_k$ and $\chi = \mu/\zeta^d_k$ or $\chi > \mu/\zeta^d_k$.

The integral G_r [see Eq. (30)] can also be rewritten in terms of $\rho(H)$ and can similarly be approximated:

$$G_r(x,e) \equiv \sum_{j;j\neq k} \int \delta^\beta_x \delta_e \frac{|U_{nj}|^{2r}}{(\lambda_k - \lambda_j)^2} \rho dH. \tag{A6}$$

The dominant contribution in this case comes from those regions of integration over U_j and λ_j which lead to $\tilde{R} \equiv \mu |U_{nk}|^2 \Sigma_j |\lambda_k - \lambda_j|^2 |U_{nj}|^2$, present in the exponent of ρ. (Note that, unlike the dominating term R in the $Q^{rs}_{mn;k}$ case, $\tilde{R}$ contains only a single component of the kth eigenfunction, namely, U_{nk}, and, the latter takes a fixed value $x/\sqrt{N}$.) Consequently, for a given Y, G_r depends on the mutual competition between μ and x. Reasoning as in the case of $Q^{rs}_{mn;k}$, G_r can be approximated as

$$G_r \approx \mu \chi_0 (N-1)^{1-r} (N - |x|^2)^r P_{11}(x)/D^2 \tag{A7}$$

with $\chi_0 = \mu^{-1}$ for $\mu |x|^2 < 1$ and $\chi_0 \sim |x|^2$ for $\mu |x|^2 > 1$.

APPENDIX B: EFFECT OF MATRIX ELEMENT PERTURBATIONS ON EIGENVALUES AND EIGENFUNCTIONS

Consider the perturbation of a Hermitian matrix H with matrix elements $H_{kl} \equiv \Sigma^2_{s=1} (i)^{s-1} H_{kl;s}$, eigenvalues λ_n, and eigenfunctions U_n, $n=1,2,\ldots,N$. By using the eigenvalue equation $\Sigma_m H_{nm} U_{mj} = \lambda_n U_{nj}$ along with the orthonormal condition on the eigenvectors, i.e., $\Sigma_j U_{nj} U^*_{mj} = \delta_{mn}$, it can be shown that

$$\frac{\partial \lambda_n}{\partial H_{kl;s}} = 2 g^{-1}_{kl} U_{kn} U_{ln},$$

$$\frac{\partial U_{nj}}{\partial H_{kl;s}} = \frac{i^{s-1}}{g_{kl}} \sum_{m\neq j} \frac{1}{\lambda_j - \lambda_m} U_{nm} [U^*_{km} U_{ln} + (-1)^{s+1} U^*_{lm} U_{kj}]. \tag{B1}$$

The details of the steps used in the derivation of Eq. (B1) can be found in [47].

The set of equations (B1) can further be used to show the following relations:

$$\sum_{k,l,s;k\leq l} \frac{\partial \lambda_n}{\partial H_{kl;s}} H_{kl;s} = \lambda_n, \tag{B2}$$

$$\sum_{k;l,s;k\leq l} \frac{\partial U_{nj}}{\partial H_{kl;s}} H_{kl;s} = 0, \tag{B3}$$

(B4)

and

$$\sum_{k,l,s;k\leq l} \frac{g_{kl}}{2} \frac{\partial^2 U_{nj}}{\partial H^2_{kl;s}} = -\sum_{m\neq j} \frac{U_{nj}}{(\lambda_j - \lambda_m)^2}, \tag{B5}$$

$$\sum_{k,l,s;k\leq l} g_{kl} \frac{\partial \lambda_i}{\partial H_{kl;s}} \frac{\partial U_{nj}}{\partial H_{kl;s}} = 0, \tag{B6}$$

$$\sum_{k,l,s;k\leq l} g_{kl} \frac{\partial U_{ni}}{\partial H_{kl;s}} \frac{\partial U_{pj}}{\partial H_{kl;s}} = -\beta \frac{U_{ni} U_{nj}}{(\lambda_i - \lambda_j)^2} (1 - \delta_{ij}) \delta_{np}, \tag{B7}$$

$$\sum_{k,l,s;k\leq l} g_{kl} \frac{\partial U_{ni}}{\partial H_{kl;s}} \frac{\partial U^*_{pj}}{\partial H_{kl;s}} = \beta \sum_{m\neq j} \frac{U_{nm} U^*_{pm}}{(\lambda_j - \lambda_m)^2} \delta_{ij}. \tag{B8}$$

[1] Y. Alhassid, Rev. Mod. Phys. **72**, 895 (2000).

[2] Supriyo Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge, U.K., 1995); D. K. Ferry and S. M. Goodnick, *Transport in Nanostructures* (Cambridge University Press, Cambridge, U.K., 1997).

[3] Ya. M. Blanter and A. D. Mirlin, Phys. Rev. E **55**, 6514 (1997); Ya. V. Fyodorov and A. D. Mirlin, Phys. Rev. B **55**, R16001 (1997); Ya. M. Blanter, A. D. Mirlin, and B. A. Muzykantskii, Phys. Rev. Lett. **78**, 2449 (1997); F. Evers and A. D. Mirlin, Phys. Rev. B **58**, 15321 (1998).

[4] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[5] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. Nunes Amaral, and H. Eugene Stanley, Phys. Rev. Lett. **83**, 1471 (1999); J. Kwapien, S. Drozdz, and P. Oswiecimka, Acta Phys. Pol. B **36**, 2423 (2005).

[6] M. A. M. de Aguiar and Y. Bar-Yam, Phys. Rev. E **71**, 016106 (2005).

[7] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).

[8] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991); C. E. Porter, *Statistical Theory of Spectra: Fluctuations* (Academic Press, New York, 1965); F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1991).

[9] T. Guhr, G. A. Muller-Groeling, and H. A. Weidenmuller, Phys. Rep. **299**, 189 (1998).

[10] Y. V. Fyodorov and A. D. Mirlin, Int. J. Mod. Phys. B **8**, 3795 (1994).A

[11] B. L. Altshuler and V. N. Prigodin, JETP Lett. **47**, 43 (1988); Sov. Phys. JETP **68**, 198 (1989).

[12] J. B. French, V. K. B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 198 (1988).

[13] V. I. Falko and K. B. Efetov, Phys. Rev. B **50**, 11267 (1994); Phys. Rev. Lett. **77**, 912 (1996).

[14] S. A. van Langen, P. W. Brouwer, and C. W. J. Beenakker, Phys. Rev. E **55**, R1 (1997); Y. Alhassid, J. N. Hormuzdiar, and N. D. Whelan, Phys. Rev. B **58**, 4866 (1998); K. Zyczkowski and G. Lenz, Z. Phys. B: Condens. Matter **82**, 299 (1991); E. Kogan and M. Kaveh, Phys. Rev. B **51**, 16400 (1995); E. Kanzieper and V. Freilikher, *ibid.* **54**, 8737 (1996).

[15] V. N. Prigodin, B. L. Altshuler, K. B. Efetov, and S. Iida, Phys. Rev. Lett. **72**, 546 (1994); V. N. Prigodin, *ibid.* **74**, 1566 (1995).

[16] K. Muller, B. Mehlig, F. Milde, and M. Schreiber, Phys. Rev. Lett. **78**, 215 (1997).

[17] M. Janssen, Phys. Rep. **295**, 1 (1998).

[18] M. V. Berry, J. Phys. A **10**, 2083 (1977); E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984); E. B. Bogomolny, Physica D **31**, 169 (1988); G. Casati and L. Molinary, Prog. Theor. Phys. Suppl. **98**, 287 (1989).

[19] P. Leboeuf and A. Voros, J. Phys. A **23**, 1765 (1990); P. Leboeuf and P. Shukla, *ibid.* **29**, 4827 (1996); P. Shukla, *ibid.* **30**, 6313 (1997).

[20] F. Izrailev, Phys. Rep. **196**, 299 (1990).

[21] V. K. B. Kota, Phys. Rep. **347**, 223 (2001); L. Benet and H. A. Weidenmuller, J. Phys. A **36**, 3569 (2003).

[22] E. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957); G. Casati, B. V. Chirikov, I. Guarneri, and F. M. Izrailev, Phys. Rev. E **48**, R1613 (1993); G. Casati, L. Molinari, and F. M. Izrailev, Phys. Rev. Lett. **64**, 1851 (1990); K. Zyczkowski, M. Lewenstein, M. Kus, and F. Izrailev, Phys. Rev. A **45**, 811 (1992).

[23] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rep. **223**, 43 (1993).

[24] E. V. Shuryak and J. J. M. Verbeerschot, Nucl. Phys. A **560**, 306 (1993); J. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993); J. Verbaarschot, *ibid.* **72**, 2531 (1994); Nucl. Phys. B **426**, 559 (1994); G. Akemann, P. H. Damgaard, U. Magnea, and S. Nishigaki, *ibid.* **487**, 721 (1997).

[25] C. W. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).

[26] H. L. Montgomery, *Analytic Number Theory, Proceedings of the Symposium on Pure Mathematics* (American Mathematical Society, Providence, RI, 1972), pp. 24, 181; A. Odlyzko, AT&T report, 1989 (unpublished); M. V. Berry, Nonlinearity **1**, 399 (1988); C. Itzykson and J.-B. Zuber, Commun. Math. Phys. **134**, 197 (1990); M. Kontsevich, *ibid.* **147**, 1 (1992); D. Aldous and P. Diaconis, Bull., New Ser., Am. Math. Soc. **36**, 413 (1999).

[27] J. Kwapien, S. Drozdz, and A. A. Ioannides, Phys. Rev. E **62**, 5557 (2000).

[28] M. S. Santhanam and P. K. Patra, Phys. Rev. E **64**, 016102 (2001).

[29] Many new applications and several other references to random matrix theory can be found by searching the cond-mat archives.

[30] E. Cartan, Abh. Math. Semin. Univ. Hambg. **7**, 116 (1935).

[31] F. Dyson, J. Math. Phys. **3**, 1191 (1962).

[32] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).

[33] T. H. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1984).

[34] A. D. Mirlin, Y. V. Fyodorov, F. M. Dittes, J. Quezada, and T. H. Seligman, Phys. Rev. E **54**, 3221 (1996).

[35] A. M. Garcia-Garcia and J. C. Osborn, Nucl. Phys. A **770**, 141 (2006).

[36] J. A. Mendez-Bermudez, T. Kottos, and D. Cohen, Phys. Rev. E **73**, 036204 (2006).

[37] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997); M. R. Zirnbauer, J. Math. Phys. **37**, 4986 (1996).

[38] J. Ginibre, J. Math. Phys. **6**, 440 (1965).

[39] J. Feinberg and A. Zee, Phys. Rev. E **59**, 6433 (1999); J. T. Chalker and B. Mehlig, Phys. Rev. Lett. **81**, 3367 (1998); Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, *ibid.* **79**, 557 (1998); Nils Lehmann and H. J. Sommers, *ibid.* **67**, 941 (1991).

[40] P. Shukla, Phys. Rev. Lett. **87**, 194102 (2001).

[41] M. V. Berry, Proc. R. Soc. London, Ser. A **400**, 299 (1985); in *Chaotic Behavior of Deterministic Systems*, edited by G. Iooss, R. H. G. Helleman, and R. Stora, Proceedings of the Les Houches Summer School of Theoretical Physics XXXVI (North-Holland, Amsterdam, 1983), p. 171; O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); G. Casati, F. Valz-Gris, and I. Guarneri, Lett. Nuovo Cimento Soc. Ital. Fis. **28**, 279 (1980).

[42] N. Dupuis and G. Montambaux, Phys. Rev. B **43**, 14390 (1991); D. Braun and G. Montambaux, *ibid.* **52**, 13903 (1995).

[43] B. L. Altshuler, V. E. Kravtsov, and I. V. Lerner, in *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (Elsevier, Amsterdam, 1991).

[44] H.-J. Sommers and S. Iida, Phys. Rev. E **49**, R2513 (1994); P. Shukla, *ibid.* **53**, 1362 (1996).

[45] M. V. Berry, J. P. Keating, and H. Schomerus, Proc. R. Soc.

London, Ser. A **456**, 1659 (2000).
[46] P. Oswiecimka, J. Kwapien, S. Drozdz, A. Z. Gorski, and R. Rak, e-print arXiv:physics/0606020.
[47] P. Shukla, Phys. Rev. E **62**, 2098 (2000).
[48] P. Shukla, J. Phys.: Condens. Matter **17**, 1653 (2005).
[49] P. Shukla, Phys. Rev. E **71**, 026226 (2005).
[50] R. Balian, Nuovo Cimento B **57**, 183 (1968).
[51] G. Casati, I. Guarneri, and F. Valz-Gris, Phys. Rev. A **30**, 1586 (1984).
[52] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990).
[53] M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
[54] M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 37 (1977).
[55] F. M. Dittes, I. Rotter, and T. H. Seligman, Phys. Lett. A **158**, 14 (1991).
[56] J. B. French and S. S. M. Wong, Phys. Lett. **33B**, 449 (1970); O. Bohigas and J. Flores, *ibid.* **34B**, 241 (1971).
[57] A. A. Arbikosov, Solid State Commun. **37**, 997 (1981).
[58] T. Terao, Phys. Rev. B **56**, 975 (1997).